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TREATISE

ON THE

ELEMENTS OF ALGEBRA.

BY

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WITH VARIOUS ADDITIONS AND IMPROVEMENTS.

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PREFACE.

When this work was first announced as preparing for publication, a near relative of the Author was to have been associated with him in its execution; but circumstances having occurred which withdrew that gentleman from mathematical pursuits, the only portions of the Work composed by him, are—the Definitions, the explanatory parts of Addition and Subtraction, and the Note at the end, containing a defence of the principle on which the Definitions are founded; and controverting the views of Dr. Peacock, and other eminent mathematicians, respecting the supposed distinction between arithmetical and symbolical Algebra.

Great pains have been taken to express the Definitions and Rules in the most simple and accurate manner, so that their meaning should be at once apparent, and that the student should, from the first, be accustomed to the most rigid mathematical exactness. As the theory of every operation is fully given, and a copious collection of examples subjoined to each Rule, this Treatise combines more theory and practice than any hitherto published: and it has therefore been necessary, in order that the Work might sell at a moderate price, to omit several subjects usually embraced in books on Algebra. It was the Author's intention to treat these subjects separately in a supplementary volume; but want of leisure has hitherto prevented him from executing this part of his plan.

The intelligent teacher will perceive that it is of great importance to introduce Equations, and questions producing Equations, at an early period of the student's progress. The examples of Simple Equations are so arranged, that a few of them may be used after the student is acquainted with Addition and Subtraction; a few more after he has learned Division;—the rest will require a knowledge of Fractions and Surds. In a regular treatise, it is necessary to bring together all that relates to the same subject, in whatever order the subjects themselves may be taken up; but, in the practice of teaching, it is not desirable to adhere strictly to this arrangement.

Many important improvements have been introduced into this edition. The chapter on the theory of numbers has been omitted, and that on the solution of simultaneous Equations considerably shortened, in order to make room for several new subjects, such as Detached Coefficients, Imaginary Quantities, Piling of Balls, &c. The chapters on Proportion and Progression, Indeterminate Coefficients, and Logarithms, have been much enlarged, and a great many other alterations introduced throughout the Work, which cannot be particularly specified, but which, with those already mentioned, will, it is confidently hoped, render it more acceptable to students of mathematics, and fully adapt it to the present advanced state of analytical science.

The Author cannot conclude without stating, that he is indebted for advice and assistance on various subjects contained in this Work to the gentleman already mentioned, and to his friend and former pupil, Mr. E. T. R. Moncrieff, of Trinity College, Dublin.

HIGH SCHOOL OF GLASGOW, SEPT. 26, 1846.

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TREATISE

ON THE

ELEMENTS OF ALGEBRA.

CHAPTER I.—FIRST PRINCIPLES.

DEFINITIONS.

ARTICLE 1. DEF. I. The sign + (read *plus*) indicates that the quantity* before which it stands is to be added to something going before: thus, 8 + 6 means that 6 is to be added to 8.

When the sign + is prefixed to a quantity standing alone, or at the beginning of a series, it indicates that there are circumstances which require that quantity to be added, if there were any quantity going before to which it could be joined: thus, +7 indicates that there are circumstances which require 7 to be added, were there any thing to which the addition could be made.

2. II. The sign — (read *minus*) indicates that the quantity to which it is prefixed is to be subtracted from something going before: thus, 8—6 means that 6 is to be taken from 8.

When the sign — is prefixed to a quantity standing alone, or at the beginning of a series, it indicates that

* Writers on Algebra generally confine the word *number* to those numerical expressions in which the Arabic figures alone occur; while they apply the term *quantity* to any symbol representing a number whether consisting of figures, or of letters, or of both combined: thus, 5, 19, are called *numbers* while 5, 19, 4*a*, may all be called *quantities*.—*See Note A at the end of the Volume.*

there are circumstances which require that quantity to be subtracted, if there were any quantity going before it: thus, —9 signifies that there are circumstances which require 9 to be subtracted, if there were any quantity from which it could be taken.

3. III. A quantity which has the sign + is called a *positive quantity*; and one which has the sign — is called a *negative quantity*. Quantities to which no sign is prefixed are understood to be positive.

4. IV. The mark = placed between two expressions indicates that they are equal: thus, $12+8=20$, indicates that the sum of 12 and 8 is equal to 20; and $15-9=6$ means that the difference of 15 and 9 is equal to 6; and $19-20=-1$ indicates that if it be required to take away 20 from 19, the thing is manifestly impossible, and that after we have taken away as much as possible, namely 19, there will still remain a unit to be subtracted, without any thing from which it can be taken. Also, $a+b=c$ shows that some number, which we call a, when increased by the addition of some other number, which we call b, becomes equal to a third number, which we denote by c.

5. *Scholium* 1. The example $19-20=-1$, will help the learner to conceive how a negative quantity may come to stand alone, that is, unconnected with any thing from which it can be taken. It will also show, that, in such a case, the sign — demands that the quantity before which it is placed be subtracted from 0;* that is to say, $-1=0-1$. And since $a-1$, or in general $a-b$, must

* For when, of the 20 to be subtracted, we have taken away as much as possible, namely 19, the + 19 will be reduced to 0, and the — 20 to —1.

be less than a, it has been said that -1 or $-b$, that is, $0-1$ or $0-b$, must be less than 0. This mode of speaking would be unintelligible and absurd, if used *absolutely;* but, in the *relative* sense in which it is always used in Algebra, it is perfectly rational and correct. Suppose the question asked, "How far is Paris north of London?" we may reply, "Paris is $2\frac{1}{2}$ degrees south of London:" but we have not given a direct answer to the question. We have, *first*, rectified the mistake on which the question was founded, by showing that it ought to have been "How far is Paris south of London?" and, *secondly*, we have given the answer to this corrected question. If we were compelled to answer in exact accordance with the original question "How far is Paris north of London?" we should say, $2\frac{1}{2}$ degrees less than nothing, that is, $-2\frac{1}{2}$ degrees. This does not mean that there exists, or can exist, any portion of space less than nothing, (for that would be a contradiction); but that the *northing* of Paris from London is less than 0: in other words, that not only is Paris nothing to the north of London, but its situation is even less northerly than those words would imply. Therefore the answer, "Paris is $-2\frac{1}{2}$ degrees to the north of London," does not merely express the *absolute* difference of latitude between the two places; but indicates, besides, that their true *relative* position is the very contrary of that which was supposed in the question. The question implies a supposition that the relation of the two places is such, that in order to find the latitude of Paris something must be added to the latitude of London: the answer exhibits this *quantity to be added* as *less than nothing*, which indicates the real relation of the two places to be such, that to find the latitude of Paris something must be subtracted from the latitude of London. In

like manner, if it be asked "How much is Edinburgh south of London?" the *direct* answer will be, $-4\frac{1}{2}$ degrees; that is to say, the question supposes the relation of the two places to be such, that in order to find the latitude of Edinburgh something must be subtracted from that of London: the answer exhibits *this quantity to be subtracted* as *less than nothing*, which indicates the real relation of the two places to be such, that to find the latitude of Edinburgh something must be added to that of London.

6. *Schol.* II. Hence, to *add* a *negative quantity* is the same as to *subtract that quantity considered absolutely*: for to speak of $-a$ being added to x, is to represent x as increased by a quantity, which is less than 0 by the quantity a. This does not mean that any quantity absolutely less than 0 can exist, but merely that the increase received by x is less than 0: in other words, this mode of expression shows that such a relation had been supposed to exist between some two things concerned in the question, as would have required a to be added; whereas the real relation has turned out to be such as requires a to be subtracted. Also, to *subtract* a *negative quantity* is the same as to *add that quantity considered absolutely*. For, to speak of $-a$ being subtracted from x is to represent x as diminished by a quantity, which is less than 0 by the quantity a: and this indicates that such a relation had been supposed to exist between some two things concerned in the question as would have required a to be subtracted; whereas the real relation has turned out to be one which requires a to be added.

7. *Schol.* III. When there is a series of quantities connected by the signs $+$, $-$, it is of no consequence in what order they are taken: thus, $7+8-3+5-9$, and

$8-3+5-9+7$, and $-3+8+5-9+7$, all give the same result, namely 8.

8. V. The mark $\times$ or a dot (.) placed between two quantities indicates that one of them is to be multiplied by the other: thus, 5×7, or 5.7, means that 5 and 7 are to be multiplied together; and $a\times x$, or $a.x$, that the numbers denoted by a and x, whatever they may be, are to be multiplied together. But when the numbers are not both expressed by the common Arabic figures, it is more usual to indicate their multiplication by writing them close together, like the letters of a word: thus, $5x$ and by are more usually written than $5\times x$ or $5.x$, and $b\times y$ or $b.y$.

9. VI. When three or more quantities are connected together by the mark of multiplication, the expression indicates that the first is to be multiplied by the second, their product by the third, that product again by the fourth, and so on: this is called *continual multiplication*: thus, $8\times6\times3$, or 8. 6. 3, signifies that 8 is to be multiplied by 6, and that the product 48 is then to be multiplied by 3, giving for the final product 144. In like manner, $4\times7\times5\times2=280$, and $3\times3\times3\times3\times3\times3=729$. Also, $a\times b\times c$, or abc, indicates that a is to be multiplied by b, and that their product is to be again multiplied by c.

10. VII. The quantities by whose multiplication a product is obtained are called its *factors*: thus 5 and 7 are the factors of 35; thus also, the factors of 30 are 6 and 5, or 10 and 3, or 5, 3, and 2; and the product abx may be considered as formed by the three factors, a, and b, and x, or by the two, ab and x.

11. *Schol.* The order in which the factors are taken does not affect the product: thus, $5\times3\times7$, $7\times3\times5$,

$5\times7\times3$, &c. all give the same result: also, $4abx$, $4xab$, $4bax$, $b4xa$, &c. are all of the same value.* But it is customary to place those factors first which are expressed in figures, and next those which are denoted by letters, arranging them in the order of the alphabet.

12. VIII. When a quantity is considered as the product of two factors, any one of these is called the *coefficient* of the other: thus, in the product $5ax$, 5 is the coefficient of ax, and $5a$ is the coefficient of x. The name *coefficient*, however, is most frequently given to a factor expressed in common Arabic figures, the other factor being a letter or letters: such for the sake of distinction, are sometimes called *numeral coefficients*. And since a may be considered as the product of 1 and a, ab as the product of 1 and ab, &c. it is plain that 1 must be understood as the coefficient of any quantity expressed by letters alone, whenever it is necessary to speak of such a quantity as having a numeral coefficient; that is to say, $a = 1a$, $ab, = 1ab$, &c.

13. IX. The mark $\div$ placed between two quantities indicates that the former is to be divided by the latter. Division is also indicated by placing the dividend directly over the divisor, and separating them by a line: thus, to show that 48 is to be divided by six, we may write either $48 \div 6$, or $\frac{48}{6}$: also, either of the expressions $a \div b$ or $\frac{a}{b}$ denotes that a is to be divided by b.

* Obvious as this proposition may appear, the most ingenious mathematicians have not been able to devise a proof of it sufficiently strict and scientific, and at the same time sufficiently simple to be of any use to the beginner.—*See Note to Art.* 36.

14. X. A *power* of a quantity is the product of two or more factors, each equal to that quantity. The powers are distinguished from one another by the number of factors they contain: that is, when there are *two* factors, the product is called the *second power*; when *three*, the *third power*; when *four*, the *fourth power*; and so on. The quantity itself from which the powers are formed, is often, for the sake of uniformity, called the *first power*; the second power is also called the *square*, and the third power the *cube*:* thus,

The 1st power of 3 is 3;
.. 2d ..... 3 .. $3\times3=9$;
.. 3d ..... 3 .. $3\times3\times3=27$;
.. 4th ..... 3 .. $3\times3\times3\times3=81$;
.. 5th ..... 3 .. $3\times3\times3\times3\times3=243$; &c.

In like manner, the 10th power of $5=5\times5\times5\times5\times5\times5\times5\times5\times5\times5=9765625$: also, xxx is the third power of x, $yyyyy$ the fifth power of y, $bbbbbbb$ the seventh power of b, and $aaaaaaaa$ the eighth power of a. These examples show that the powers of a letter will be expressed by writing that letter as often as there are units in the number by which the power is distinguished; that is, *twice* for the second power, *three times* for the third power, &c. But as this method would be very inconvenient for the higher powers, it has been agreed not to write the letter more than once, but to place over it

* The application of the terms *square* and *cube* to algebraic quantities is very improper: it has arisen from the manner in which geometrical magnitudes are expressed in numbers; for whatever number expresses the linear units in a given straight line, the second power of that number will express the number of corresponding superficial units in the area of the square of that line, and its third power the number of solid units in the cube of that line.

and towards the right hand a number indicating how often it ought to be written; that is, how often it is to be used as a factor. Thus, instead of writing xxx for the third power of x, we write x^3, where the number 3 indicates that we are to understand x as written three times; or, which is the same thing, that we are to conceive of a product formed by three factors, each equal to x. A number thus employed to indicate how many factors a power contains is called the *index* or *exponent* of the power. Accordingly.

The 1st power of a is $a = a^1$, and its index is 1;
.. 2d ..... a.. $aa = a^2$, ... 2;
.. 3d ..... a.. $aaa = a^3$, ... 3;
.. 4th ..... a.. $aaaa = a^4$, ... 4;
.. 5th ..... a.. $aaaaa = a^5$, ... 5; &c.

Indices are chiefly used with letters, but sometimes also with common numbers: thus, $3^4 = 81$, $4^3 = 64$.

The process of finding any power of a quantity is called *involution*; and the quantity is said to be *raised to* that power.

15. XI. A *root* of a quantity is some other quantity of which the given quantity is a power: thus, $8^2 = 64$, therefore 8 is a root of 64; $4^3 = 64$, therefore 4 is also a root of 64; $2^6 = 64$, therefore 2 is another root of 64.

The different roots of a quantity are distinguished by numbers, indicating the power to which each root must be raised, in order to become equal to the given quantity; that is, the root of which the given quantity is the second power is called the *second root*, that of which it is the third power is called the *third root*, that of which it is the fourth power is called the *fourth root*, &c. Thus in the above example, the second root of 64 is 8, the third root 4,

and the sixth root 2. In like manner, the fourth root of 81 is 3, the fifth root of 1024 is 4, &c.

The process of finding any root of a quantity is called *evolution*; and the quantity is said to have that root *extracted.*

The second root is also called the *square* root, and the third the *cube* root (Art. 14),

16. XII. The mark $\surd$ prefixed to a quantity indicates that a root of that quantity is to be extracted: this mark is called the *radical sign.* The number by which the root is distinguished is written over the radical sign to the left, and is called the *index of the root:* thus, $\sqrt[2]{25}$ means the second root of 25; $\sqrt[3]{27}$, the third root of 27. In using the radical sign to denote the second root, the index 2 is usually omitted: thus, for the second root of 25, we may either write $\sqrt[2]{25}$, or $\surd 25$. The roots of any number, a, will be expressed as follows:

The 2d root of a is $\sqrt[2]{a}$, or $\surd a$;
.. 3d ..... a .. $\sqrt[3]{a}$;
.. 4th ..... a .. $\sqrt[4]{a}$;
.. 5th ..... a .. $\sqrt[5]{a}$; &c.

17. XIII. Whatever number of factors a quantity contains besides its coefficient, of so many *dimensions* it is said to be: thus, bc, $7ax$, $3x^2$, are quantities of two dimensions, because each of them contains two factors in addition to its coefficient. In like manner, xyz, $5ax^2$, are quantities of three dimensions; xy^3, $12a^2x^2$, are quantities of four dimensions, and so on. On the same principle, a, $15x$, are quantities of one dimension.

Instead of "one dimension," "two dimensions," &c. the terms *first order* or *degree*, *second order* or *degree*, &c. are often used: thus, $7ax$ is said to be of the *second order* or *second degree*; $5ax^2$, of the *third order* or *third*

degree, &c. Sometimes a letter is considered as belonging to the coefficient, and then it is not reckoned in estimating the dimensions or degree of the quantity: thus, in the example $5ax^2$, if we consider $5a$ as the coefficient, the quantity will be of two dimensions only.

18. XIV. An expression which does not consist of parts connected by either of the signs + or —, is called a *simple* or *monomial quantity:* thus, 7 and x are simple quantities. In like manner, $6 \times 3 \times 4$ is a simple quantity: also, $12ax$, and $\frac{bc}{x}$.

19. XV. An expression which consists of two or more simple quantities connected by the signs +, —, is called a *compound quantity* and the simple quantities of which it is composed are called its *terms:* thus, $6 + 3 + 4$ is a compound quantity, and its terms are 6, 3, and 4: also, $12 + a - x$, and $3y + 12bc - \frac{bx}{y}$, are compound quantities; the terms of the former being 12, a, and x, and those of the latter, $3y$, $12bc$, and $\frac{bx}{y}$.

20. XVI. A compound quantity consisting of two terms is called a *binomial*. A binomial whose second term is negative is sometimes called a *residual* quantity. A quantity consisting of more terms than two is called a *multinomial* or *polynomial* quantity. The names *trinomial* and *quadrinomial* are sometimes given to quantities consisting of three and four terms

21. XVII. A parenthesis enclosing an expression is called a *vinculum*, and denotes that the sign or index on the outside of it affects the result of all the operations indicated by the several signs and indices within it. Thus,

18—(5+7) means that 7 is first to be added to 5, and then the sum taken from 18, which leaves 6; without the vinculum, 18—5+7 would mean that 5 is to be taken from 18, and 7 added to the remainder, which would give 20: also, 16—(15—9)=10, but 16—15—9=—8. In like manner, (12+5)×7, or 7(12+5)=119, but 12+5×7 =47. And $(10+3)^2=169$, but $10+3^2=19$; and $\sqrt{(25+11)}=6$, but $\sqrt{25}+11=16$. Also, $(xy)^2$ would mean that x is to be multiplied by y, and the second power of the product taken; whereas xy^2 would mean that the second power of y was to be taken, and x multiplied by it; or, supposing $x=5$ and $y=3$, $(5\times3)^2=225$, but $5\times3^2=45$.

The learner must carefully observe, that in all such cases the sign on the outside of the vinculum affects no one of the terms within the vinculum, each of which has, besides, a sign of its own; that is, in the expressions 18—(5+7), and 16—(15—9), the terms 5 and 15 are understood to have the sign + (Art. 3).

A bar drawn over the quantities is also used as a vinculum: the bar is sometimes placed vertically: thus,

$\overline{a+b-c}\times d$, and $+\begin{matrix}a\\b\\-c\end{matrix}\Big|\begin{matrix}d\\\\\end{matrix}$, mean the same as $(a+b-c)\times d$.

Brackets [ ], and the brace { }, are used instead of the parenthesis when we have occasion to place one vinculum outside of another; as, $[(a+b)x+(a-b)x]^2$.

22. XVIII. *Like quantities* are those in which the parts expressed by letters are the same: thus, $54ay$, $6ay$, and $-19ay$, are like quantities. But $7ay$, and $7a^2y$, are *unlike*; because the factor a occurs twice in the latter, and only once in the former, for (Art. 14) $7a^2y=7aay$. In the same manner, $17a^2x$, and $5xa^2$, are like quantities (Art. 11); but $4a^2x$, and $6a^3x$, are unlike.

23. *Schol.* When two or more like quantities are connected by the signs $+$, $-$, they may be incorporated into one term, by performing on their coefficients the operations indicated by their signs, and to the result subjoining the common letter: thus, $+5x+3x+7x=15x$: for the first term contains five times x to be added, the next contains three times x to be added, and the next contains seven times x to be added, making in all fifteen times x to be added, that is to say, $+15x$. In like manner, $-5x-3x-7x=-15x$; for, in this case, the $x$es are all to be subtracted. Also, $6ab+7ab-5ab-3ab+4ab=+9ab$: for $6ab$ with $7ab$ added will make, as above, $+13ab$; and $5ab$ subtracted will leave $+8ab$; $3ab$ further subtracted will leave $+5ab$; and $4ab$ added to this will give $+9ab$. In like manner, $5ax^2+2ax^2-8ax^2-4ax^2+3ax^2=-2ax^2$; and $10xy-12xy+xy=-xy$. Since (Art. 7) it is indifferent in what order the terms are taken, we may first unite all the positive quantities into one term, and all the negative into another, and then subtract the latter from the former: thus, $5ax^2+2ax^2-8ax^2-4ax^2+3ax^2=10ax^2-12ax^2=-2ax^2$.

24. XIX. The mark $>$ or $<$, placed between two quantities, denotes that the one to which the opening of the lines is turned is greater than the other. Hence $a>b$ means that a is greater than b; and $x<y$, that x is less than y.

25. XX. When it is not known which of two quantities is the greater, the mark $\backsim$ placed between them indicates that the less is to be subtracted from the greater: thus, $x \backsim y$ indicates that if, of the two quantities, x be the greater, y is to be subtracted from x; but if y be the greater, x is to be subtracted from y; that is, the expression $x \backsim y$ is equivalent to $x-y$ when $x>y$, and to $y-x$ when $x<y$.

PRAXIS ON THE DEFINITIONS.

26. The conversion of literal into numerical expressions will be found a useful exercise on the meaning of the symbols and operations of Algebra. The letters are supposed to denote particular numbers, which are substituted for the letters, and the same operations performed upon them that are performed upon the letters.

Thus, let it be required to find what the expression

$$a+b^2+ac-abx+\frac{abc}{5dx}-2axy$$

will be in numbers, when

$$a=4,\ b=5,\ c=3,\ d=2,\ x=1, \text{ and } y=0.$$

Substituting the numbers for the letters, the expression becomes

$$4+5\times5+4\times3-4\times5\times1+\frac{4\times5\times3}{5\times2\times1}-2\times4\times1\times0;$$

or, performing the multiplications,

$$4+25+12-20+\frac{60}{10}-0;$$

by contracting, this is reduced to

$$41-20+6, \text{ or } 27,$$

which is the value of the expression on the above supposition respecting the letters.

EXERCISES.

1. What is $a+b-c$, when $a=5$, $b=4$, and $c=6$?

Ans. 3.

2. What is the same expression, when $a=7$, $b=2$, $c=8$? *Ans.* 1.

3. What is the same expression, when $a=10$, $b=6$, $c=12$? *Ans.* 4.

4. What is the same expression, when $a=9$, $b=4$, $c=13$? *Ans.* 0.

5. What is the same expression, when $a=8$, $b=3$, $c=12$? *Ans.*—1.

6. What is the same expression, when $a=1$, $b=2$, $c=6$? *Ans.* —3.

7. What is $3a-2c+b$, when a, b, and c, are successively, 4, 5, 6; 3, 2, 4; 8, 6, 15; and 1, 9, 7? *Ans.* 5; 3; 0; and—2.

8. If a, b, c, are 4, 5, 6; and 5, 7, 6, what is $a(c-b)$? *Ans.* 4 and —5

9. If $a=7$, $b=2$, what is $a+b$? *Ans.* 9.

10. What is ab, on the same supposition? *Ans.* 14.

11. If $a=7$, $b=2$, $x=9$, what is $a+b+x$? *Ans.* 18.

12. What is abx, on the same supposition? *Ans.* 126.

13. If $a=1$, $b=2$, $x=5$, what is abx? *Ans.* 10.

14. Find the difference between abx, and $a+b+x$, when $a=4$, $b=\frac{1}{2}$, $x=3$; and when $a=5$, $b=7$, $x=12$? *Ans.* $1\frac{1}{2}$, and 396.

15. What must a and b be in whole numbers, in order that ab may be 6,? *Ans.* 3, and 2, or 6 and 1.

16. What is $x^2+2xy+y^2$, when $x=4$, $y=6$? *Ans.* 100.

17. What is $x^2-2xy+y^2$, when $x=4$, $y=6$? *Ans.* 4.

18. What is $n(n-1)\ (n-2)\ (n-3)$, when $n=4$; and when $n=10$? *Ans.* 24, and 5040.

19. What is $s\ (s-a)\ (s-b)\ (s-c)$, when $s=\frac{1}{2}(a+b+c)$, and $a=4$, $b=7$, $c=9$; and again, when $a=12$, $b=8$, $c=6$? *Ans.* 180 and 455.

20. Find the difference between a^2 and $2a$, when a is 5? *Ans.* 15.

21. Find the difference between a^3 and $3a$, when a is 5, and when a is 10? *Ans.* 110, 970.

22. What is $abc+2c-3d+2ef$, when $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$? *Ans.* 0

23. What is $a^6+b^4-c^2-d+e+4abcdef$, on the same supposition? *Ans.* 9.

24. What is $5ab+\frac{c+d-e}{2a}-2bf-2e$, on the same supposition? *Ans.* 1.

25. What is $\sqrt{6bc}+\sqrt[3]{(bd)}+ab+3$, on the same supposition? *Ans.* 13.

26. What is $\sqrt{(4e+b+c)}-\sqrt[3]{bc+2}-2d$, on the same supposition? *Ans.* —5.

27. If $a=6$, $b=5$, $c=4$, $x=1$, $y=0$, what is $2a^2-5b-\frac{bc}{4x}-8a+y^2$? *Ans.* —6.

28. What is $x^{10}+5y^4+\frac{abcx}{3cx}-\frac{abcxy}{5y^3}-2b$, on the same supposition? *Ans.* 1.

29. What is $3\sqrt{c}+2a\sqrt{(2a+b-x)}$, on the same supposition? *Ans.* 54.

30. What is $\frac{2b+c}{3a-c}-\frac{\sqrt{5b}+3\sqrt{c+x}}{2a+c}$, on the same supposition? *Ans.* $\frac{1}{4}$.

31. What is $[(a+b)x+(a-b)x]^2$ on the same supposition? *Ans.* 144.

32. What is $n(n-1).(n-2).(n-3).(n-4)$, if n be 1, 2, 3, 4, 5, and 6, successively?

Ans. 0,0,0,0,120,720.

33. What must a and x be, so that $(a+x)^2$ may be 100?

Ans. Any two numbers whose sum is 10.

34. What number must x be equal to, in order that abx may become ab? *Ans.* 1.

35. Find the difference between $4x-4$ and $4x \div 4$, when $x=3$; and also when $x=10$?

Ans. 5, and 26.

36. What is $3ax+5a^2$, if x be taken equal to a?

Ans. $8a^2$.

37. What is $3abx-3b^3+4a^3-2a^2x$, when b and x become each equal to a? *Ans.* $2a^3$.

38. What is $4xy-3bd+5x^2+2b^2-3y^2$, when $y=x$, and $d=b$? *Ans.* $6x^2-b^2$.

39. What does the expression $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ become when $x=5$? *Ans.* $10=10$.

40. What does $\sqrt{x}-\frac{8}{x}=\frac{7}{\sqrt{x}-2}$ become when $x=16$?

Ans. $3\frac{1}{2}=3\frac{1}{2}$.

41. If $x=\frac{m+n}{m-n}$, what is x, when $m=\frac{1}{2}$, $n=\frac{2}{3}$?

Ans. 9.

42. If $y=\frac{3}{1+x}+\frac{3}{1-x}$, what is y, when $\frac{1}{2}$ is put for x?

Ans. $y=8$.

43. Let $u=\frac{x}{y}-\sqrt{\frac{1+x}{1-y}}$, and $x=\frac{1}{4}$, $y=\frac{1}{3}$; what is u?

Ans. 0.

44. If $y=(\frac{3\sqrt{x}}{5}-2)\ (\frac{1}{x-5})$, what is y, when $x=25$; and when $x=49$? *Ans.* $\frac{1}{20}$ *in both cases.*

45. If $y=5\sqrt{(62+3x)}-\frac{1}{2}\sqrt{(95\frac{8}{9}-5x)}$, what is y, when $x=6\frac{1}{3}$? *Ans.* 41.

46. Find what $x^2+2xy+y^2$ is in numbers, when $x=5$, $y=3$; when $x=10$, $y=3$,; and when $x=\frac{1}{2}$, $y=2$?

47. What is $x^2-2xy+y^2$, when $x=7$, $y=13$?

48. When $a=3$, what is the difference between a^4 and $4a$?

49. Find the difference between a^5 and $5a$, when $a=3$, and between a^2 and $2a$, if a be successively $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 3, and 4.

50. What is $a^4+2ab+b^4$, when $a=1$, $b=2$; and when $a=\frac{1}{2}$, $b=\frac{1}{4}$?

51. If $a=6$, $b=5$, $c=4$, $x=1$, $y=0$, what are the expressions $\frac{2a+3c}{6x+4y}+\frac{4bc}{\sqrt{(2ac+c^2)}}$, and $\sqrt{(2a^2-\sqrt{4c^2})}+bxy+\sqrt{(bc-c)}+x^5$?

52. Let s be as in Ex. 19, and $a=15$, $b=14$, $c=13$, what will $\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ then be?

53. Find the difference between $6abc-2ab$, and $6abc\div 2ab$, when a, b, c, are 3, 5, and 6 respectively?

54. Find the difference between each pair of the following expressions; $(9+3)\times 8$ and $9+3\times 8$; $\sqrt{(49+15)}$, and $\sqrt{49}+15$; $(8+4)^2$ and $8+4^2$?

55. Find the difference between each pair of the following expressions, a being 6, and b 8; $(a+b)^2$ and a^2+b^2; $a+b$ and $\sqrt{(a^2+b^2)}$; $(a+b)^3$ and $a+b^3$; $\sqrt{(a^2+b^2)}$ and $\sqrt{a^2}+\sqrt{b^2}$; $5(a+b)$ and $5a+b$.

56. It will appear hereafter, that, whatever values a and b may have, $\frac{a^2-b^2}{a-b}=\frac{a^3+b^3}{a^2+b^2-ab}$. Illustrate the truth of this by taking any numbers at pleasure for a and b.

Thus, let $a=4$, $b=2$; then

$$\frac{a^2-b^2}{a-b}=\frac{16-4}{2}=6;$$

and

$$\frac{a^3+b^3}{a^2+b^2-ab}=\frac{64+8}{16+4-8}=\frac{72}{12}=6.$$

Again, let $a=5$, $b=3$; then

$$\frac{a^2-b^2}{a-b}=\frac{25-9}{2}=8;$$

and

$$\frac{a^3+b^3}{a^2+b^2-ab}=\frac{125+27}{25+9-15}=\frac{152}{19}=8;$$

and the two expressions will be always found equal, whatever numbers are taken for a and b.

The following expressions will also be shown to be true hereafter: illustrate their truth in the same manner.

57. $(a+b)\ (a+b)=a^2+2ab+b^2$.

58. $(x+y)\ (x-y)=x^2-y^2$.

59. $(n-m)\ (n-m)=n^2-nm+m^2$.

60. $(a+b)+(a-b)=2a$.

61. $(a+b)-(a-b)=2b$.

62. $(x-4)\ (x+7)=x^2+3x-28$.

63. A person is asked, "How much is it past 12 o'clock?" He looks at his watch, and observes that it wants 5 minutes of 12; how must he give a direct answer to the question?

64. How much is a merchant worth when his debts amount to £3,000, and his entire property is only £2,500?

65. If a quantity of goods cost £750, and be sold for £730, how much has the merchant gained?

66. How far is the Cape of Good Hope north of the equator?

AXIOMS.

27. Ax. I. If equals be added to equals, the sums are equal; but if equals be added to unequals, the sums are unequal: thus, if $a=x$, then $a+b=x+b$, or if $a=x$ and $b=y$, then $a+b=x+y$; but if $a>x$, then $a+b>x+b$; and if $a<x$, then $a+b<x+b$.

28. II. If equals be subtracted from equals, the differences are equal; but if equals be subtracted from unequals, or unequals from equals, the differences are unequal: thus, if $a=x$, then $a-b=x-b$; or if $a=x$ and $b=y$, then $a-b=x-y$: but if $a>x$, then $a-b>x-b$; and if $a<x$, then $a-b<x-b$, &c.

29. *Corollary* 1. From Ax. I. it follows, that if equals be multiplied by equals, the products are equal; but if equals be multiplied by unequals, the products are unequal: for multiplication is nothing else than a number of successive additions.

30. *Cor.* II. Hence, if equals be divided by equals, the quotients are equal: for, by the nature of division, each dividend is a product, of which the divisor and the quotient are the factors; and since the divisors are equal, the remaining factors, the quotients, must be equal, else (*Cor.* 1.) the products, the dividends, could not be equal. For a similar reason, if equals be divided by unequals, or unequals by equals, the quotients are unequal.

31. *Cor.* III. Again, equal quantities raised to the same power are equal, and equal quantities raised to different powers are unequal: thus, if $a=x$, then $a^n=x^n$; but if $a>x$, then $a^n>x^n$; and if $a<x$, then $a^n<x^n$; for involution is only a succession of multiplications. Also, the same roots of equal quantities are equal; for if it were otherwise, the given quantities, being obtained by raising those roots to the same power, could not be equal.

CHAP. II.—FUNDAMENTAL OPERATIONS.

ADDITION.

ART. 32. Connect the several quantities to be added by their respective signs, incorporating like quantities. (Art. 23).

First, Let the quantity to be added be simple. If it be required to add a to x; this, from the nature of algebraic notation, can only be done by placing the sign + between them, which gives

$$x+a$$

for the sum. And if it be required to add $-a$ to x; this is done (Art. 6) by writing

$$x-a.$$

Secondly, Let the quantity to be added be compound. It is plain that to add a compound quantity is the same as to add its separate terms; that is,

$$x+(a+b)=x+a+b;$$

and

$x+(a-b)=x+a+(-b)$; that is, (Art. 6) $x+a-b$.

Thirdly, Let some of the quantities to be added be like. Thus, to $x+3a$ add $y+5a$: this would be done by writing $x+3a+y+5a$, which (Art. 7 and 23) is the same as $x+y+8a$.

It is often convenient to place the quantities to be added over one another: thus the last example may be written as under.

$$\begin{array}{r} x+3a \\ y+5a \\ \hline x+y+8a \end{array}$$

EXERCISES.

1. $4x^2y^2-5xy$
$5ax+x^2y^2$
$7x^2y^2+3xy$
$-2ax-5x^2y^2$
$7x^2y^2-2xy+3ax$

2. $4a-2x+5b$
$-3x+2a-2y$
$3b-5a+7x$
$-4a+2x-6b$
$-3a+2b+4x-2y$

3. $2ax-3b^2y$
$3ax-2b^2y$
$ax-b^2y$
$5ax-4b^2y$
$ax-b^2y$
$12ax-11b^2y$

4. $4a+2x$
$-3a-3x$
$a+7x$
$2a-4x$
$-5a+x$
$3a-6x$
$2a-3x$

5. $a+b$
$b-e$
$c-d$
$d-e$
$b-c$
$a+3b-2e$

6. $4x(a+b)-1-3\sqrt{x}$
$4\sqrt{x}+2x(a+b)-7$
$8-\sqrt{x}-5x(a+b)$
$3x(a+b)-5\sqrt{x}-5$
$4x(a+b)-5\sqrt{x}-5$

7. $ax+by+5x^2$
$bx-3y+2x^2$
$-3x^2-dx+dy$
$ax+bx-dx+by+dy-3y+4x^2$

or,

$(a+b-d)x+(b+d-3)y+4x^2$*

* This answer, in connexion with Art. 23, shows that to multiply the sum of any number of quantities by another quantity is the same as to multiply those quantities separately, and then to add the products.

8. $-3a-4b+5c$
$2b-4c+6ey$
$7c-8d-9ey$
$3b-a-3d$

9. $a+b-c$
$a-b+2c$
$-3a+2b-3c$
$2a-b+2c$

10. $3a+3x^2-b^3$
$-5a-x^3+12b^3$
$7a-7x^2+13b^3$
$-4a-4x^2-20b^3$

11. $-12ax-5b$
$3ax-5$

12. $-8+a$
4

13. $9-6x$
$3x+x^2$

14. Add together $x+a-c$, $3b+2d$, $-a+4x$, $-3x+5a$.

15. Add together $(a+b)\sqrt{x}-d\sqrt{(x^2-y^2)}-mnp$, $2ax+8mnp+3(a+b)\sqrt{x}$, $p-3d\sqrt{(x^2-y^2)}+2bx$, $-5mnp+3p-2(a+b)\sqrt{x}$.

16. Add together $3\sqrt{x}-\sqrt{(a+b)}$, $-2\sqrt{x}+\sqrt{(a+b)}$, $\sqrt{x}-3\sqrt{(a+b)}$, $4\sqrt{x}+2\sqrt{(a+b)}$.

17. Add together $3a\sqrt{x}-2\sqrt[3]{y}-5+ax^2$, $ax+11-a\sqrt{x}+3\sqrt[3]{y}$, $-2ax^2-4\sqrt[3]{y}+4ax$, $-8+2a\sqrt{x}-3ax+2\sqrt[3]{y}$, $m\sqrt{x}-bx+6+4ax^2$

18. Add together a and b.

19. ..... $a+b$ and $a-b$.

20. ..... x and x^2.

21. ..... n and $\sqrt{n}$.

22. ..... $7a$ and $-8a$.

23. ..... 1 and -2.

24. ..... a and $-a$.

25. ..... $3d$ and $4c$.

26. ..... $6x$ and $-3y$.

27. ..... $\sqrt{x}$ and $-2\sqrt[3]{x}$.

28. Add together $-a^2$ and $-a$.
29. ..... a^2x^3 and ax^3.
30. ..... $\frac{1}{2}a-\frac{1}{2}b$ and $\frac{1}{2}a+\frac{1}{2}b$.
31. ..... $a+b$ and $a-2b$.
32. ..... $x+p-y$ and $y-x$.
33. ..... $6+7x$ and $-3x+x^2$.
34. ..... a^2+ax and x^2-ax.
35. ..... $1+\sqrt{x}$ and $1-\sqrt{x}$.
36. ..... $-11x^3$ and $-5x^2$.
37. ..... $2a+b$ and $5a^2-b$.
38. ..... a^2 and a^3.
39. ..... b^3 and $3b$.
40. ..... $6a^n+5\sqrt[n]{x}-\frac{4}{3}y$ and $-4\sqrt[n]{x}+\frac{1}{3}y-2a^n$.
41. ..... $5a$ and $-11a$; also $-5a$ and $11a$.
42. ..... $5axy$ and $7ax$.
43. ..... -9 and 5.
44. ..... $2x(a+b)$ and $3x(a+b)$.
45. ..... $-19-x$ and x.
46. ..... $x+a$ and $-a$; also $a-1$ and 1.
47. ..... $-(a+x)y$ and $2(a+x)y$.
48. ..... $5x+3b-a$ and $a-2b$.
49. ..... $4b^4-1+2\sqrt{x}$ and $-3b^4+5-3\sqrt{x}$.
50. ..... $3a^m-b^n+3x^p$, $2a^m-3b^n-x^p$, and $a^m+4b^n-x^p$.
51. ..... $3a^2-5a+1$, $7a^2+2a-4$, and $13-a^2-4a$.
52. ..... $a+b-c$, $a+c-b$, and $c+b-a$.
53. ..... $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$.
54. ..... $6axy$ and $-4xay$.
55. ..... $2a^n+1$ and $3a^{n+1}$.
56. ..... $a+b+c-d$, $a+b+d-c$, $a+c+d-b$, and $b+c+d-a$.

33. Change the signs of the subtrahend, and incorporate like quantities. (Art. 23).

First. Let the quantity to be subtracted be a simple quantity.

From x
subtract a.

This can only be done by writing $x-a$; that is to say,

$$x-(+a)=x-a.$$

Again, from x
subtract $-a$

This (Art. 6) is the same as to add a, which is done by writing $x+a$; that is,

$$x-(-a)=x+a$$

Secondly. Let the quantity to be subtracted be a compound quantity. It is plain that to subtract the compound quantity is the same as to subtract its terms in succession; that is,

$$x-(a+b)=x-a-b:$$

and

$x-(a-b)=x-a-(-b)$; that is, (Art. 6) $x-a+b$*

* The same may be shown otherwise, as follows:

First, from x
subtract $a+b$.

If a be taken away, giving $x-a$, too little has been taken away by the quantity b; therefore b must further be taken away, which gives $x-a-b$.

Secondly, from x
subtract $a-b$.

If a be taken away, giving $x-a$, too much has been taken away by the quantity b; therefore the remainder $x-a$ is too small by b: hence b must be added, which gives $x-a+b$.

EXAMPLES.

1. From $7a^2-4ax+11$ take $-3a^2+2ax+5$.

Here, changing the signs of the subtrahend, or conceiving them changed, and incorporating the like terms, we have $10a^2-6ax+6$, for the difference required.

$$\begin{array}{r} 7a^2-4ax+11 \\ -3a^2+2ax+5 \\ \hline 10a^2-6ax+6 \end{array}$$

2. From $5a-6x$ subtract $2a-3x$. Here the required difference is $5a-6x-(2a-3x) = 5a-6x-2a+3x = 3a-3x=3(a-x)$

EXERCISES.

1. $$\begin{array}{r} -6x \\ -11x-5 \\ \hline 5x+5=5(x+1) \end{array}$$

2. $$\begin{array}{r} b-c+d+8 \\ 19-b^2+c+d \\ \hline -11+b+b^2-2c \end{array}$$

3. $$\begin{array}{r} 2a\sqrt{(x^2+y)}-5a^2x+11 \\ 10-5a\sqrt{(x^2+y)}-8a^2x \\ \hline 7a\sqrt{(x^2+y)}+3a^2x+1 \end{array}$$

4. $$\begin{array}{r} ax^2+bx+cxy-d \\ ex^2-fx+gxy-h \\ \hline x^2(a-e)+x(b+f)+xy(c-g)-d+h \end{array}$$

5. $$\begin{array}{r} 4ax^2y \\ 3axy \\ \hline 4ax^2y-3axy \end{array}$$

6. $$\begin{array}{r} -7a \\ 3a \\ \hline -10a \end{array}$$

7. $$\begin{array}{r} pmx \\ pm \\ \hline pmx-pm \end{array}$$

8. $$\begin{array}{r} \sqrt[4]{x} \\ \sqrt[3]{x} \\ \hline \sqrt[4]{x}-\sqrt[3]{x} \end{array}$$

In the above general examples, the lower line, or subtrahend, contains only two terms, yet it exemplifies all the variety of signs that can occur in any compound quantity. The upper line, or minuend, is no less general; because, whatever number of terms it contained, they would be all set down unchanged in the difference, the change being made on the subtrahend only.

9. $-9a$ / $4a$

10. $-9a$ / $-4a$

11. $9a$ / $-4a$

12. $9a$ / $4a$

13. $a+b$ / $a-b$

14. $a-b$ / $a+b$

15. b / c

16. a / $-a$

17. $-a$ / $-a$

18. x^3 / x^2

19. ab / a

20. 17 / $5-a-3b$

21. $2(x+y)-3(m-n)$ / $x+y-4(m-n)$

22. a^2+ax+x^2 / x^2+a^2-ax

23. $b(x+y)^2$ / $(b+c)(x+y)^2$ / $-c(x+y)^2$

24. $-ax^2-(m-n)\sqrt{(a+b)}$ / $-(3+a)x^2+(2m-3n)\sqrt{(a+b)}$ / $3x^2-(3m+2n)\sqrt{(a+b)}$

ANSWERS.

25. $1-(1-a)$ ..... a.
26. $1-(1+a)$ ..... $-a$
27. $1-(-1-a)$ ..... $2+a$
28. $-1-(1-a)$ ..... $a-2$
29. $-(1-a)$ ..... $a-1$
30. $-(1+a)$ ..... $-1-a$
31. $1+a-(1+a)$ 0
32. $1-a-(1-a)$ 0
33. $1-a-(1+a)$ ..... $-2a$
34. $-(1-a)-(1+a)$ ..... -2

ANSWERS.

35. $a-1-(-1-a)$ ..... $2a$

36. $1+a-(a-1)$ 2

37. $a-b-(a+b)$ ..... $-2b$

38. $a+b-(a-b)$ ..... $2b$

39. $\frac{1}{2}(a+b)-\frac{1}{2}(a-b)$ ..... b

40. $x-7-(2x-5-x)$ ..... $-2(x+1)$

41. $a-(b-c+x)+(b-x-2b)$ ..... $a+2b+c$

42. $a-\{a-(a-a-x)\}$ x

43. $x^2+2xy+y^2-[x^2+xy-y^2-(2xy-x^2-y^2)]$.

Ans. $y^2+3xy-x^2$

44. $2a\sqrt{(x-y)}-bxy-a\sqrt{(x-y)}-bxy$.

Ans. $a\sqrt{(x-y)}$.

45. $c(a+x)-(a+c)(a+x)$. *Ans.* $-a(a+x)$

46. $(a+b)(x+y)-(a-b)(x+y)$. *Ans.* $2b(x+y)$.

47. $4a-\sqrt{x}-(3a+2\sqrt{x})$.

48. $a+b-[b+(a-b)]$

49. $a-b+c-d-(a+b-c+d)$.

50. $4a^2-3b+7x-(6x+2b^2-4b+2a^2+a)$.

51. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$.

52. $1-\sqrt{x}-(\sqrt{x}+1)$.

53. $1-[1-(1-1-x)]$

54. From $4x$ ..... take 4.

55. .... $3\sqrt{x}$ ..... $2\sqrt{x}$.

56. .... a^2 ..... a.

57. .... 0 ..... $-b$.

58. .... $3ab$ ..... $3a$.

59. .... $2x$ ..... x^2.

60. .... 1 ..... 2.

61. .... $4a$ ..... $6a$.

62 .... $4a$ ..... $-6a$.

63. ....$-4a$ ..... $-6a$.

64. ....$-4a$ ..... $6a$.

65. .... 5 ..... 6.
66. .... $\frac{5}{4}x$ ..... $-\frac{1}{4}x$.
67. .... $-\frac{5}{4}x$ ..... $\frac{1}{4}x$.
68. .... $-\frac{5}{4}x$ ..... $-\frac{1}{4}x$.
69. .... $\frac{5}{4}x$ ..... $\frac{1}{4}x$.
70. .... 11 ..... 14.
71. .... $1+a$ ..... $a+1$.
72. .... $1-a$ ..... $a-1$.
73. .... 0 ..... $-(a-b)$.
74. .... 7 ..... -8.
75. .... $\frac{1}{2}a-\frac{1}{2}b$ ..... $\frac{1}{2}b-\frac{1}{2}a$
76. .... -11 ..... 6.
77. .... $b^2+2ab+a^2$ ..... $b^2-2ab+a^2$
78. .... $\sqrt{x}(a-1)$ ..... $a-1$.
79. .... $\sqrt[5]{x}$ ..... $\sqrt[5]{x}$.
80. .... $19-x$ ..... $-x$.
81. .... $a-1+b$ ..... -1.
82. .... $x+a$ ..... a.
83. .... $n-1$ ..... 1.
84. .... $5a-3$ ..... -3.
85. .... $-a$ ..... $-b-4$.

From Ex. 39, 38, and 13, and Ex. 53, 35, 30, and 19 of Addition, in connexion with Ex. 60, and 61 of the Praxis, it follows, that if the sum and difference of two unequal quantities be given, *the greater will be found by adding half the sum to half the difference, and the less, by taking half the difference from half the sum.* Thus if 86 and 28 be respectively the sum and difference of two numbers, the numbers are 57 and 29; and if the half sum, and half difference of two numbers be 32 and 14, the numbers are 46 and 18.

34. PROP. I. If the signs of the factors are the same, the product is positive; and if the signs of the factors are different, the product is negative.

First, Let it be required to multiply $+a$ by $+b$. This denotes that $+a$ is to be repeated as often as there are units in b; that is, that $+a$ is to be added b times. Now the sum of any number of positive quantities is positive (Art. 23); therefore the sum is $+ba$: hence $(+a)\times(+b)=+ba$, or $+ab$ (Art. 11).

$+a$
$+a$
$+a$
$+a$
$+a$
$+$ &c. as far as b times.
$+ba$

Secondly, Let it be required to multiply $-a$ by $+b$. This denotes that $-a$ is to be added as often as there are units in b; and since the sum of any number of negative quantities is negative (Art. 23,) this sum will be negative; that is to say, it will be $-ba$: hence $(-a)\times(+b)=-ba$.

$-a$
$-a$
$-a$
$-a$
$-a$
$-$ &c. to b times.
$-ba$

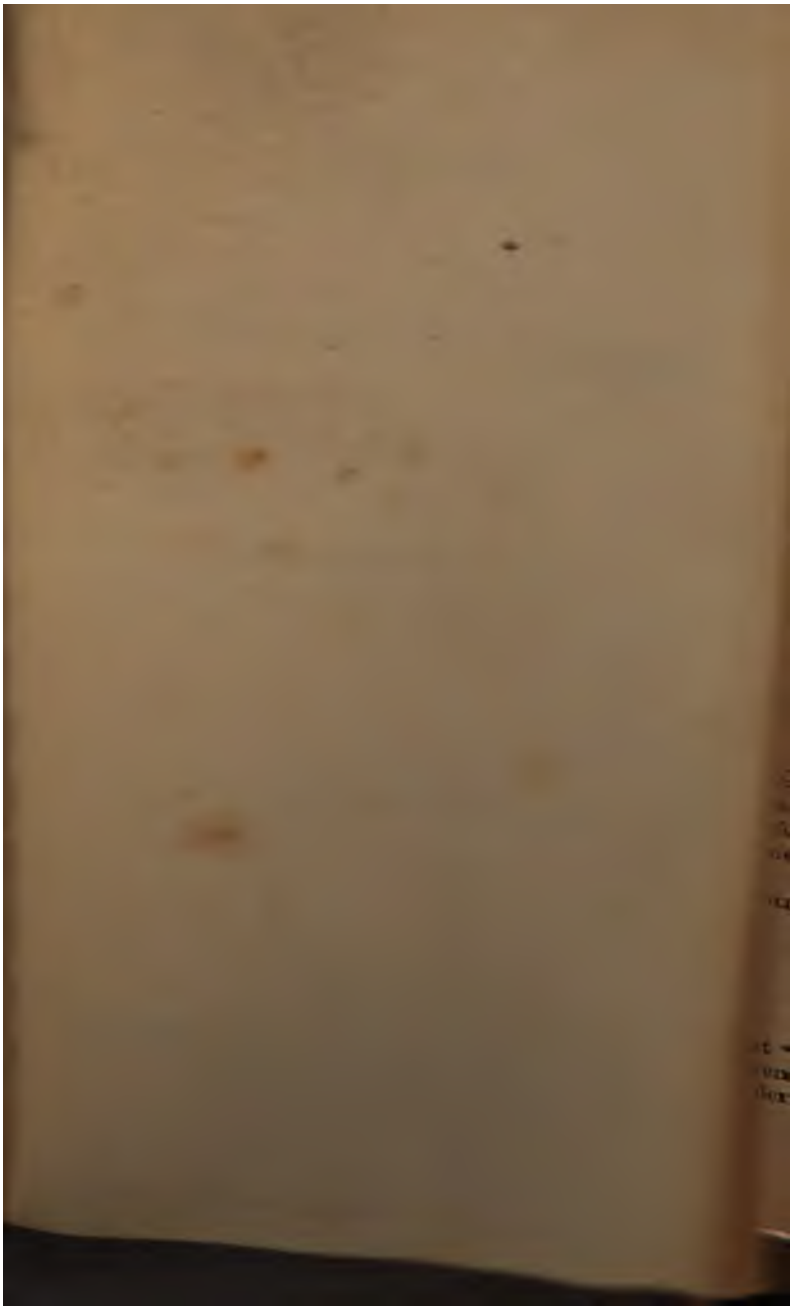
Thirdly, Let it be required to multiply $+a$ by $-b$. This denotes that $+a$ is to be *repeated negatively* * as often as there are units in b (the negative repetition being indicated by the sign of b;) in other words, a is to be subtracted b times (Art. 6,) or a is to be added b times, and the sum subtracted: hence $(+a)\times(-b)=-ba$.

$+a$
$+a$
$+a$
$+a$
$+a$
$+$ &c. to b times.
$-(+ba)=-ba$.

* That is, added less than 0 times. This means that circum-

46.

697.



Fourthly, Let it be required to multiply $-a$ by $-b$. This denotes that $-a$ is to be repeated negatively b times; that is, $-a$ is to be subtracted b times; or, in other words, $-a$ is to be added b times, and the sum subtracted: but (Art. 23) the sum is $-ba$: and this sum being subtracted will be $+ba$: hence $(-a)\times(-b)=+ba$.*

$$\begin{array}{ll} & -a \\ & -a \\ & -a \\ & -a \\ & -\&c.\text{ to} \\ & b\text{ times.} \\ \hline -(-ba)= & +ba \end{array}$$

stances had been supposed to exist which would have required as many additions of a as there are units in b; whereas the real circumstances have turned out to be such as to require the same number of subtractions.

* This proposition may be demonstrated as follows after the method of multiplying compound quantities has been learned.

Let it be required to multiply $a-b$ by $c-d$.

Here $a-b$ is first to be multiplied by c: the product of $+a$ by $+c$ is $+ac$, as shown above; and the product of b by c is bc. But since a is greater than $a-b$ by b, the product ac is too great by bc; bc must therefore be subtracted: hence the product is $+ac-bc$.

Again, $a-b$ is to be multiplied by $-d$; but without regarding the sign of d, multiply $a-b$ by d, and the product is $ad-bd$, as before. Now, since c is greater than $c-d$ by d, in multiplying by c, a multiplier too great by d has been used: hence the product of $a-b$ by c is greater than the true product by the product of $a-b$ by d; the product of $a-b$ by d, or $ad-bd$, must therefore be taken from that of $a-b$ by c, or $ac-bc$: this gives

$$ac-bc-(+ad-bd)=ac-bc-ad+bd.$$

The process will, then, stand thus:

$$\begin{array}{r} a-b \\ c-d \\ \hline ac-bc \\ \quad -ad+bd \\ \hline ac-bc-ad+bd. \end{array}$$

This proof may be more briefly given. To multiply $a-b$ by $c-d$, put $a-b=p$, then the product is $p\,(c-d)=pc-pd$: or, putting for p its equal,

$$\begin{aligned} (a-b)\,(c-d) &= (a-b)\,c-(a-b)\,d \\ &= ac-bc-(ad-bd) \\ &= ac-bc-ad+bd. \end{aligned}$$

35. II. Powers of the same quantity are multiplied by adding their indices.

For (Art. 14) $a^2 \times a^3 = aa \times aaa = aaaaa = a^5 = a^{3+2}$, and $a^7 \times a^4 = aaaaaaa \times aaaa = aaaaaaaaaaa = a^{11} = a^{7+4}$: and, in general, if a^n be multiplied by a^m, the product must contain as many factors, each equal to a, as are contained in a^n and a^m together; that is, the number of such factors must be $n+m$, and therefore the product must be a^{n+m}

36. Case I.—*When both multiplier and multiplicand are simple quantities.* To the product of the numerical coefficients annex the product of the letters.*

The following numerical illustration will make this proposition still more evident. Let it be required to multiply 14—8 by 12—5. This is obviously the same as to multiply 6 by 7; the product is therefore 42: but for the sake of illustration, let us proceed as in the foregoing general example. The product of 14 and 12 is 168, which is too great by 12 times 8, or 96: hence the true product is 168—96 or 72, which is more simply formed; for 14—8=6, and 6×12=72. Again, the product of 14—8 by 5, is 70—40, or 6 × 5 = 30. But since the first multiplier, 12, is too great by the second, 5, the first product, 72, is too great by the second, 30; 30 must therefore be subtracted, and 72—30=42, the same as was at first obtained by following the simple and natural process.

14—8
12—5
168—96=72
subtract 70—40=30
168—96—70+40=72—30=42

The steps of the process may be more briefly put down thus:

$$\begin{aligned}(14—8)\,(12—5) &= (14—8)\,12—(14—8)\,5\\ &= 168—96—(70—40)\\ &= 168—96—70+40\\ &= 72—30=42.\end{aligned}$$

* The following method of establishing the proposition that "the order of the factors does not affect the product," is given by Legendre, "Théorie des Nombres," p. 1; it is one of the demonstrations adverted to in the note to Art. 11.

EXAMPLES.

1. Multiply $5\,bc$ by $3\,a$.

Here (Art. 8) the product is $5\,bc \times 3\,a$, or (Art. 11) $5 \times 3 \times bc \times a$; that is, $15\,bca$, or (same Art.) $15\,abc$.

$$\begin{array}{r} 5bc \\ 3a \\ \hline 15\,abc \end{array}$$

2. $\begin{array}{r} 2xy \\ 6ab \\ \hline 12\,abxy \end{array}$

3. $\begin{array}{r} 4a^2 \\ -3a^4 \\ \hline -12a^6 \end{array}$

4. $\begin{array}{r} 5a^2x^3 \\ 2a^3xy \\ \hline 10a^5x^4y \end{array}$

5. $\begin{array}{r} -2\sqrt{x} \\ -3\sqrt{x} \\ \hline 6x \end{array}$

6. $\begin{array}{r} -7ax \\ 3bc \\ \hline -21\,abcx \end{array}$

7. $\begin{array}{r} 2a^2\sqrt{y} \\ 5a^3\sqrt{y} \\ \hline 10a^5y \end{array}$

37. CASE II.—*When the multiplier is simple, and the multiplicand compound.* Multiply the several terms of the compound quantity by the simple multiplier, and connect the products by their proper signs.

This rule depends upon the principle in the note to p. 21.

Let there be a rectangular figure whose adjacent sides are divided into any number, a and b, of equal parts; then, if parallels be drawn from the points of division, the figure will be divided into rows of equal squares. Reckoning in one way, there will be a rows, each containing b squares, and the whole number will therefore be ab. But if we reckon in the other way, there will be b rows, each containing a squares, and the whole number will be ba. Now, the number of squares is always the same; therefore, $ab=ba$.

Of this demonstration he observes, that it appears to him "à-la-fois claire, générale, et exacte; cependant si on lui reproche d'être fondée sur des notions d'étendue étrangères à la science des nombres, voici comment on pourra y suppléer:" and he proceeds to give another demonstration founded on the assumption that, $a \times 1 = 1 \times a$, to which, however, it may be objected that this is inadmissible as an axiom, and requires proof nearly as much as that $ab=ba$. Various other proofs have been given; but they are equally unsatisfactory. No systematic proof, however, could render more obvious the truth of a proposition, so simple, and so nearly self-evident.

EXAMPLES.

1. $3a+4b-6x$
$2y$
$6ay+8by-12xy$

2. $2a^3+4a^2x^2-3b^2y$
$-3a^2x$
$-6a^5x-12a^4x^3+9a^2b^2xy$

3. $2a+3x\sqrt{y}-\sqrt{y}$
$2\sqrt{y}$
$4a\sqrt{y}+6xy-2y$

4. $a-\sqrt{x}-1$
$a\sqrt{x}$
$a^2\sqrt{x}-ax-a\sqrt{x}$

38. CASE III.—*When both multiplier and multiplicand are compound quantities.* Multiply every term of the multiplicand by each term of the multiplier, and add the several products thus obtained.

For from the note in p. 21, and from Art. 11. it appears, that to multiply any quantity by the sum of any number of quantities, is the same as to multiply by those quantities separately, and then to add the products.

When different powers of the same quantity are given, the terms should be arranged according to the indices of those powers, in order that, in the partial products, like terms, when such occur, may be, as much as possible, under like terms.

EXAMPLES.

1 Multiply $a+b$ by $a+b$.

$a+b$
$a+b$

Multiply $a+b$ by a ..... a^2+ab
..... $a+b$ by b ..... $ab+b^2$

Add the partial products $a^2+2ab+b^2$

2. Multiply $5x^4-4x^3+3x^2-3x+1$ by $3x^2-2x-1$.

$$\begin{array}{l} 5x^4-4x^3+3x^2-3x+1 \\ 3x^2-2x-1 \\ \hline \text{Multiply each term by } 3x^2 \quad 15x^6-12x^5+9x^4-9x^5+3x^2 \\ \text{..... by } -2x \qquad -10x^5+8x^4-6x^3+6x^2-2x \\ \text{..... by } -1 \qquad -5x^4+4x^3-3x^2+3x-1 \\ \hline \text{Add the partial products} \quad 15x^6-22x^5+12x^4-11x^3+6x^2+x-1 \end{array}$$

3. Required the product of $3a^5+2a^4x-a^2x^3$ and $x^5-ax^4+2a^3x^2$.

Here the terms being arranged according to the descending powers of a, and ascending powers of x, the process is as follows :—

$$\begin{array}{l} 3a^5+2a^4x-a^2x^3 \\ 2a^3x^2-ax^4+x^5 \\ \hline 6a^8x^2+4a^7x^3-2a^5x^5 \\ \qquad -3a^6x^4-2a^5x^5+a^3x^7 \\ \qquad\qquad 3a^5x^5+2a^4x^6-a^2x^8 \\ \hline 6a^8x^2+4a^7x^3-3a^6x^4-a^5x^5+2a^4x^6+a^3x^7-a^2x^8 \end{array}$$

4. $$\begin{array}{l} 5-2\sqrt{3} \\ 4+3\sqrt{3} \\ \hline 20-8\sqrt{3} \\ \qquad 15\sqrt{3}-18 \\ \hline 2+7\sqrt{3} \end{array}$$

5. $$\begin{array}{l} b+x \\ b-x \\ \hline b^2+bx \\ \qquad -bx-x^2 \\ \hline b^2-x^2 \end{array}$$

6. $$\begin{array}{l} a^n-2a^{n-2}x^2+3a^{n-4}x^4-4a^{n-6}x^6 \\ a^2-3x^2 \\ \hline a^{n+2}-2a^nx^2+3a_n{}^{-2}x^4-4a^{n-4}x^6 \\ \qquad -3a^nx^2+6a^{n-2}x^4-9a^{n-4}x^6+12a^{n-6}x^8 \\ \hline a^{n+2}-5a^nx^2+9a^{n-2}x^4-13a^{n-4}x^6+12a^{n-6}x^8 \end{array}$$

7 $4a^5c^2-3a^2bc^2+2c^3$

$4a^5c^2+2a^2bc^2-c^3$

$16a^{10}c^4-12a^7bc^4+8a^5c^5$

$+8a^7bc^4-6a^4b^2c^4+4a^2bc^5$

$-4a^5c^5+3a^2bc^5-2c^6$

$16a^{10}c^4-4a^7bc^4+4a^5c^5-6a^4b^2c^4+a^2bc^5-2c^6$

8. What will $(a+b)x$ become when $a+c$ is put for x. Here $(a+b)x$ becomes $(a+b)(a+c)$, or $a^2+ab+ac+bc$.

9. What will x^2 become when x is changed into $x+h$. Here x^2 becomes $(x+h)^2$ or $(x+h)(x+h)$ that is $x^2+2xh+h^2$.*

39. SCHOL. The literal parts in the factors of a required product, are often so related, that the law according to which the terms of the product will succeed one another is manifest, without actual multiplication. Thus in Ex. 2, last Art., the indices of x in each factor decrease by a

* Since, by the nature of the process, the products of all binomial factors will be formed as in Exs. 1 and 9, we may conclude, in general, that, *the square of a binomial is equal to the squares of its terms and twice their product.* The same will hold when the terms are not both positive, for $(-a-b)^2=a^2+2ab+b^2=(a+b)^2$, and $(a-b)^2$ or $(b-a)^2=a^2-2ab+b^2$. The two forms may be combined in one expression by means of the double sign; thus: $(a\pm b)^2=a^2\pm 2ab+b^2$. By applying this formula we may often dispense with actual multiplication; thus we have directly, $(3a+2b)^2=9a^2+12ab+4b^2$; $(5-y^2)^2=25-10y^2+y^4$; $39^2=(30+9)^2=30^2+2\times 30.9+9^2=900+540+81$. We may square a trinomial in the same manner, by taking two terms as one; thus $\overline{(a+b)+c}^2=(a+b)^2+2(a+b)c+c^2=a^2+2ab+b^2+2ac+2bc+c^2$:—and the same principle may be extended to quadrinomials—*See Euc.* II., 4, and Exs. 57, 59, Art. 26.

Again, from Ex. 5 we see that *the product of the sum and difference of two quantities is equal to the difference of their squares;* and conversely, that *the difference of the squares of two quantities is equal to the product of the sum and difference of those quantities.* Thus for the product of $(4a+3x)\times(4a-3x)$ we have directly, without multiplication, $16a^2-9x^2$; and the factors of $25a^4-16b^6$ are $5a^2+4b^3$ and $5a^2-4b^3$. See Ex. 58, Art. 26.

unit from term to term, and the same takes place in the product: also in Exs. 3 and 6, the indices of a diminish, and those of x increase, from term to term in the factors and product. The same is shown by Ex. 1, while Ex. 7 does not present a like symmetry, either in the factors or product. A little consideration will show that in general the products will follow the law of the factors. In such cases, then, the literal part of the product will be known at once, and multiplication will be necessary merely to determine the coefficients, This is done by writing out in order the several coefficients, detached from the letters, with their respective signs, and operating upon them according to the Rule of Art. 38, the type, or form, of the process being precisely the same as if they were connected with the several letters. To the result so obtained the several powers are then subjoined. It is manifest, that if in adding the partial products, zero is obtained as the coefficient in any term, the continuity in the powers is interrupted, and that term will be wanting from the complete product. It is also plain, that if, in either factor, the powers do not succeed one another in regular order, one or more powers being wanting, zero must be put for the coefficient of such terms, in writing out the detached coefficients. This being premised, multiplication by the method of detached coefficients, will present no difficulty.

EXAMPLES.

1. Required the work of Ex. 2, last Art. by this method.

$$
\begin{array}{r}
5 \quad -4 \quad +3 \quad -3 \quad +1 \\
3 \quad -2 \quad -1 \\
\hline
15 \quad -12 \quad +9 \quad -9 \quad +3 \\
\quad -10 \quad +8 \quad -6 \quad +6 \quad -2 \\
\quad\quad -5 \quad +4 \quad -3 \quad +3 \quad -1 \\
\hline
15 \quad -22 \quad +12 \quad -11 \quad +6 \quad +1 \quad -1
\end{array}
$$

Now the literal part of the first term being $x^4 \times x^2 = x^6$, and the powers forming a descending series, we have for the complete product,

$$15x^6 - 22x^5 + 12x^4 - 11x^3 + 6x^2 + x - 1$$

2. Required the product as in Ex. 3, last Art., by this method.

In this Ex. the third term, namely a^3x^2, is wanting to complete the regular succession of powers in the first factor, and the second term, namely a^2x^3, in the second factor, and therefore in writing out the detached coefficients, zero must be put for the coefficients of these terms; the work will then be as on the margin. The double sign $\pm$ is prefixed to the coefficient 0, since it is doubtful what sign the wanting term would have had. Now the literal part of the first term being $a^5 \times a^3x^2 = a^8x^2$, and the terms being arranged according to the descending powers of a, and ascending powers of x, the rest of the literal part is easily supplied; the complete product then is

3	+2	±0	−1			
2	±0	−1	+1			
6	+4	±0	−2			
	±0	−3	−2	±0	+1	
				3	+2	±0	−1
6	+4	−3	−1	+2	+1	−1

$$6a^8x^2 + 4a^7x^3 - 3a^6x^4 - a^5x^5 + 2a^4x^6 + a^3x^7 - a^2x^8.$$

3. Required the product as in Ex. 6, by this method.

Here the indices of a in either factor diminish, while those of x increase, by 2 from term to term; and the same will hold in the product; the work therefore will proceed as on the margin. To the coefficients thus found the letters may then be annexed, as in the answer to Ex. 6. The question might also be worked by putting

1	−2	+3	−4	
1	−3			
1	−2	+3	−4	
	−3	+6	−9	+12
1	−5	+9	−13	+12

0 for the coefficients of the intermediate terms, those, namely, in which the given powers would have the indices odd.

4. Find the product of $2a^3x-2ax^3-3x^4$ and $3a^2-2ax+3x^2$

Here, in the first factor the term containing the second powers does not occur; and 0 is put for its coefficient. In the result 0 is found as the coefficient of the third and fifth terms; these therefore are wanting and the literal part of the first term being $a^3x \times a^2=a^5x$, the complete product will be

$$6a^5x-4a^4x^2 \quad -5a^3x^4 \quad -9x^6;$$

the spaces being left to indicate the terms that are wanting.

$$\begin{array}{llll} 2 & \pm 0 & -2 & -3 \\ 3 & -2 & +3 & \\ \hline 6 & \pm 0 & -6 & -9 \\ & -4 & \pm 0 & +4 \quad +6 \\ & & +6 & \pm 0 \quad -6 \quad -9 \\ \hline 6 & -4 & \pm 0 & -5 \quad \pm 0 \quad -9 \end{array}$$

Many of the exercises that follow will afford practice on this method. It possesses considerable advantages over the common method when the terms contain two letters only; if there are more than two it may still be employed to some extent, but less conveniently.

EXERCISES.

1. Multiply a by b.
2. ..... x by np.
3. ..... a^3 by a^5.
4. ..... a^5 by x^5.
5. ..... $7x$ by $8xy$.
6. ..... abc by def.
7. ..... 7 by b.
8. ..... $8x$ by $5x^5$.
9. ..... $2a^2$ by $5a^3$.
10. ..... a by -1.
11. ..... -11 by $-a$.

12. Multiply mn by pq.
13. ..... $4x$ by $\frac{1}{2}a$.
14. ..... $\frac{3}{4}b^2$ by $\frac{8}{3}ab^3$.
15. ..... a^n by a^p.
16. ..... $4b^{n-1}$ by $3b^{n+1}$.
17. ..... $-2b^3$ by $-4b^4$.
18. ..... b^{n-1} by b^2.
19. ..... axy by $-ax$.
20. ..... a^{n-2} by a^{n+1}.
21. ..... x^n by x^3.
22. ..... $-4c$ by $12d$.
23. ..... $a^n x^n$ by a^2x^3.
24. ..... $\sqrt{a}$ by itself.
25. ..... $3\sqrt{5}$ by $4\sqrt{5}$.
26. ..... $\frac{1}{2}a$ by $\frac{1}{2}a$.
27. ..... $2\sqrt{b}$ by $\sqrt{b}$.
28. ..... $4\sqrt{x}$ by $5\sqrt{x}$.
29. ..... $\frac{1}{3}a^2x^3$ by $\frac{1}{3}a^3x^5$.
30. ..... $4a$ by $\frac{1}{2}a$.
31. ..... $2\sqrt{ax}$ by $4b$.
32. ..... $a^5x^2y^4$ by $a^{10}x^3y$.
33. ..... $4x-2b+3d^2$ by $2x$.
34. ..... $-7a^2+4abx-3d+8$ by $3a^3$.
35. ..... $a+bx+edy$ by ab.
36. ..... $7a-4ax+2by$ by $-4a^3x$.
37. ..... $4ax-2b^2y+7ad-5y$ by $3ab$.
38. ..... $4\sqrt{x}-3x^3y+5\sqrt[3]{x}$ by $2ay$.
39. ..... $2y^3-3xy+12ax-3$ by $\frac{1}{8}xy$.
40. ..... $y-3x+2az$ by my.
41. ..... $b^2y^3z-ay^3d+2y^5$ by $-bx$.
42. ..... $a+x$ by $a+x$.
43. ..... $a+b$ by $c+d$.
44. ..... $3-v$ by $3-v$.

45. Multiply $6+a$ by $6-a$.
46. ..... $4-a$ by $5+b$.
47. ..... $r-3$ by $2r-4$.
48. ..... $n-4$ by $n+4$.
49. ..... $4a-\frac{1}{2}$ by $4a-\frac{1}{2}$.
50. ..... $m+n$ by $m+p$.
51. ..... $a+x$ by $a-x$.
52. ..... $n-p$ by $n-p$.
53. ..... $n-\frac{1}{4}$ by $n+\frac{1}{4}$.
54. ..... $\frac{1}{2}a-1$ by $\frac{1}{2}a-1$.
55. ..... $3b+2y$ by $3b+2y$.
56. ..... $5x^2-3y^3$ by $5x^2-3y^3$.
57. ..... a^4+b by a^4+b.
58. ..... $x+y$ by $x+y$.
59. ..... $x+3$ by $x+5$.
60. ..... $x-4$ by $x-11$.
61. ..... $x+7$ by $x-4$.
62. ..... $x+9$ by $x-10$.
63. ..... a^n+b^n by $a-b$.
64. ..... a^n+b^n by a^n+b^n.
65. ..... $1-\sqrt{2}$ by $1+\sqrt{2}$.
66. ..... $6a^2+10ax-4x^2$ by $4a-2x$.
67. ..... $a^m+a^{m-1}x+a^{m-2}x^2+a^{m-3}x^3$ by $a+x$.
68. ..... a^m-2b^n by $2a^m-3b^n$.
69. ..... x^2+4x+8 by x^2-4x+8.

Ans. x^4+64.

70. ..... $a^3+a^2b+ab^2+b^3$ by $a-b$.

Ans. a^4-b^4.

71. ..... $x^2+2xy+y^2$ by $x+y$.

Ans. $x^3+3x^2y+3xy^2+y^3$.

72. ..... $a^2+a^4+a^6$ by a^2-1. *Ans.* a^8-a^2.

73. ..... $1+2x+3x^2+4x^3+5x^4$ by $1-2x+x^2$

Ans. $1-6x^5+5x^6$

74. Find the continued product of $x-1$, $x+2$, $x+3$, and $x-4$. *Ans.* $x^4-15x^2-10x+24$.

75. Find the continued product of $x-5$, $x+6$, $x-7$, $x+8$. *Ans.* $x^4+2x^3-85x^2-86x+1680$.

76. Multiply $a+b+c$, $-a+b+c$, $a-b+c$, and $a+b-c$ continually together.

Ans. $-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2.)$

77. Multiply $a^4-2a^3b+16b^4-8ab^3+4a^2b^2$ by $a+2b$.

Ans. a^5+32b^5.

78. Multiply $5ab+3ac-4bc$ by $7ab-18ac+2bc$.

Ans. $35a^2b^2-69a^2bc-18ab^2c-54a^2c^2+78abc^2-8b^2c^2$.

79. Multiply $9+2\sqrt{10}$ by $9-2\sqrt{10}$. *Ans.* 41.

80. Multiply $2a^3x^2-3a^4x^4+4a^5x^6-5a^6x^8$ by $ax^2+2a^2x^4-3a^3x^6$.

Ans. $2a^4x^4+a^5x^6-8a^6x^8+12a^7x^{10}-22a^8x^{12}+15a^9x^{14}$.

81. Multiply $2x^3y^3+3xy^5-5x^4y^2$ by $7x^2y^3-x^3y^2+2x^4y$.

Ans. $21x^5y^6-3x^4y^7+20x^5y^6-37x^6y^5+9x^7y^4-10x^8y^3$.

82. Multiply $7a^3x-3a^2x^2+2ax^3$ by $3a^2x^2+ax^3-x^4$.

Ans. $21a^5x^3-2a^4x^4-4a^3x^5+5a^2x^6-2ax^7$.

83. Multiply $2x^3+4x^2+8x+16$ by $3x-6$.

84. Multiply $4x^3-29x+4x^2+21$ by $2x-3$.

85. Required the product of $a^5x^3+2a^3x^2+ax$ and $a^6x^3-4a^4x^2+a^2x$.

86. Multiply $4v^4+5v^5-3v^6$ by $v-3v^2+2v^3$

87. Multiply v^3-3v^2+4 by v^3+3v-4.

88. Multiply $5a^3-3ab^2-2b^3$ by $3a^4+2a^2b^2-5ab^3$.

40. PROP. I. When the divisor and dividend have the same sign, the quotient is positive; when they have different signs, the quotient is negative.

For, by the nature of division, the quotient must be such, that when multiplied by the divisor, it shall reproduce the dividend; therefore,

First, If $+ab$ be divided by $+a$, the quotient must be $+b$, in order that the product $+ab$ may be reproduced (Art. 34.)

Secondly, If $-ab$ be divided by $+a$, the quotient must be $-b$, in order that the product $-ab$ may be reproduced.

Thirdly, If $+ab$ be divided by $-a$, the quotient must be $-b$, in order that the product $+ab$ may be reproduced.

Fourthly, If $-ab$ be divided by $-a$, the quotient must be $+b$, in order that the product $-ab$ may be reproduced.

41. II. A power of a quantity is divided by any power of the same quantity, by subtracting the index of the divisor from the index of the dividend.

For (Art. 14, 40) $a^5 \div a^3 = aaaaa \div aaa = aa = a^2 = a^{5-3}$; and $a^{11} \div a^7 = aaaaaaaaaaa \div aaaaaaa = aaaa = a^4 = a^{11-7}$; and, in general, if a^n be divided by a^m, the number of factors, each equal to a, in the quotient, will be found by subtracting the number of such factors in a^m from the number of such factors in a^n; that is, the number of factors will be $n-m$, therefore the quotient is a^{n-m}.

42. CASE I.—*When both divisor and dividend are simple quantities.* Take for the quotient such a quantity as, multiplied by the divisor, will reproduce the dividend.

EXAMPLES.

1. Divide $20bxy$ by $5x$.

Here $20bxy$ being a product, and one of its factors $5x$, the other, by the nature of division, must be $4by$. The quotient, therefore, consists of the quotient of the numerical coefficients, with those letters annexed which are not common to the divisor and dividend.

$$5x)20bxy(4by$$
$$\underline{20bxy}$$
$$0$$

2. Divide $12a^4bn^3x^5$ by $-3a^2nx^4$.

$$-3a^2nx^4)12a^4bn^3x^5(-4a^2bn^2x.$$

3. Divide $-9b^3x^4$ by $-4ab^2x^3$.

Here -4 is contained $2\frac{1}{4}$ times in -9, and ab^2x^3 in b^3x^4 is contained $\frac{bx}{a}$ times, the common factors being expunged.

$$-4ab^2x^3)-9b^3x^4(2\tfrac{1}{4}\frac{bx}{a}, \text{ or } \frac{9bx}{4a}$$
$$\underline{-9b^3x^4}$$
$$0, \text{ or}$$
$$\frac{-9b^3x^4}{-4ab^2x^3}=\frac{9bx}{4a}$$

4. Divide $6bx^2y^3$ by $2abdx^2$.

$$2abdx^2)6bx^2y^3(\frac{3y^3}{ad}, \text{or} \frac{6bx^2y^3}{2abdx^2}=\frac{y^3}{ad}\times\frac{2bx^2}{2bx^2}=3\frac{y^3}{ad}$$

5. Divide a by $\sqrt{a}$.

Here, as in other cases, the quotient must be such as to produce a when multiplied by $\sqrt{a}$; it must therefore be $\sqrt{a}$.

$$\sqrt{a})a(\sqrt{a}$$
$$\underline{a}$$
$$0$$

6. Divide x^n by x^{n-1}.

$$x^n\div x^{n-1}=x^{n-(n-1)}=x^{n-n+1}=x^1=x.$$

7. Divide $5b^3x^4y$ by $7abx^3y$. *Ans.* $\frac{5b^2x}{7a}$.

8. Divide $-12a^nb^2$ by $-4a^2b^2$. *Ans.* $3a^{n-2}$.
9. Divide $18x^{n-1}$ by $-9x^{n-m}$. *Ans.* $-2x^{m-1}$.
10. Divide $-x^{2n}$ by $-x^n$. *Ans.* x^n.

43. CASE. II.—*When the divisor is simple, and the dividend compound.* Divide every term of the dividend by the divisor, and connect the quotients by their proper signs.

EXAMPLES.

1. Divide $12ab^2c-32bcx^2-4abcy-4bc$ by $4bc$.

$$4bc)12ab^2c-32bcx^2-4abcy-4bc$$
$$3ab - 8x^2 - ay - 1.$$

2. Divide $4a^2b-12b+16ax-4x^2-14a^3b^2$ by $-4a$.

$$-4a)4a^2b-12b+16ax-4x^2-14a^3b^2$$
$$-ab+3\frac{b}{a}-4x+\frac{x^2}{a}+3\tfrac{1}{2}a^2b^2.$$

3. Divide $a^n-a^{n+1}x+a^{n+2}x^2-a^{n+3}x^3+a^{n+4}x^4$ by a^n.

$$a^n)a^n-a^{n+1}x+a^{n+2}x^2-a^{n+3}x^3+a^{n+4}x^4$$
$$1 - ax + a^2x^2 - a^3x^3 + a^4x^4$$

4. Divide $-15n^4x^3+9n^3x^5-12n^2x^2+3n^2x^5$ by $-3n^2x^2$.

$$-3n^2x^2)-15n^4x^3+9n^3x^5-12n^2x^2+3n^2x^5$$
$$5n^2x-3nx^3+4-x^3.$$

44. CASE III.—*When both divisor and dividend are compound quantities.* (1) Place the quantities in one line, as in division in Arithmetic; (2) arrange, if possible, both divisor and dividend according to the powers of some letter common to both, so that its

highest power may stand first, its next highest power second, and so on; (3) divide the first term of the dividend by the first term of the divisor; the quantity found is the first term of the quotient; (4) multiply this term into the divisor, and, (5) subtract the product from the dividend; (6) consider the remainder, if there be any, as a new dividend, and proceed as before; and (7) if a remainder do not disappear, indicate its division by the divisor, and annex the result to the quotient.

EXAMPLES.

1. Divide $2ax+x^2+a^2$ by $a+x$.

Arranging the terms according to the powers of a, (Art. 38,) and dividing a^2 by a, we find a for the first term of the quotient; (4) multiplying the divisor by this term a, we have a^2+ax, which being subtracted (5) from the dividend, leaves the remainder $ax+x^2$ for the new dividend. Again, dividing (6) the first term ax of this remainder by the first term a of the divisor, we have x for the second term of the quotient, the sign being $+$ because a and ax have the same sign; multiplying again, we have $ax+x^2$ to subtract, which leaves no remainder: the complete quotient is therefore $a+x$.

$$\begin{array}{r|l} a+x)a^2+2ax+x^2(a+x \\ \underline{a^2+ax} \\ \quad ax+x^2 \\ \quad \underline{ax+x^2} \\ \quad 0 \end{array}$$

By this process, the dividend or given product, is analysed into several partial products, and as the whole dividend is made up of these, the divisor is obviously contained in the whole as often as it is contained in all the parts; these products are subtracted from the dividend in succession, first, the product of the divisor $a+x$ by a, the first term of the quotient, and then the product of the

divisor $a+x$ by x, the second term of the quotient; so that, in all, we take from the dividend the product of $a+x$ by $a+x$, without leaving any remainder: since therefore the quotient and divisor reproduce the dividend without remainder, this quotient is complete, and the operation is finished. If there be a remainder, the quotient is incomplete; and if the remainder do not disappear, on continuing the process, the quotient is interminable, and there is no series of terms which, multiplied by the divisor, can reproduce the dividend.* (*See* CHAP. III. *last Art.*)

2. Divide x^4+a^4 by $x+a$.

Here the divisor is put to the right of the dividend and the quotient below it, in order to save room.† Then $x^4 \div x = x^3$ the first term of the quotient, and $x^3(x+a)=x^4+ax^3$;

$$
\begin{array}{l}
x^4+a^4 \,(x+a \\
\underline{x^4+ax^3} \quad x^3-ax^2+a^2x-a^3+\dfrac{2a^4}{x+a} \\
\quad -ax^3+a^4 \\
\quad \underline{-ax^3-a^2x^2} \\
\qquad\quad a^2x^2+a^4 \\
\qquad\quad \underline{a^2x^2+a^3x} \\
\qquad\qquad -a^3x+a^4 \\
\qquad\qquad \underline{-a^3x-a^4} \\
\qquad\qquad\qquad 2a^4
\end{array}
$$

* We might have arranged the terms according to the powers of x, thus $x+a)x^2+2xa+a^2$,
and the operation would give $x+a$ for quotient.

If this mode of arranging the terms be disregarded, we shall sometimes find the complete quotient after a greater number of operations than are necessary: in most cases, however, the quotient will be interminable. The following arrangement of the terms of the last example will make this more plain:

Here one operation more than was necessary is gone through, in consequence of the arrangement, and the quotient is reducible to $a+x$ (*See* CHAP. III. *last Art.*)

$$
\begin{array}{l}
a+x)2ax+a^2+x^2(2x+a-x \text{ or } a+x \\
\quad \underline{2ax+2x^2} \\
\quad a^2-x^2 \\
\quad \underline{a^2+ax} \\
\qquad -ax-x^2 \\
\qquad \underline{-ax-x^2} \\
\qquad\quad 0
\end{array}
$$

† By this arrangement, moreover, the multiplication necessary

this being snbtracted, gives $-ax^3+a^4$ for a new dividend. Again, $-ax^3 \div x = -ax^2$, the second term of the quotient; this is multiplied into the divisor, and so the work proceeds till the remainder $2a^4$ is found. This term does not contain x and the work terminates.—(*See* CHAP. III., *last Art.*) The remainder $2a^4$ with the divisor subscribed is then annexed to the quotient. In verifying the work by multiplying the divisor by the quotient, this remainder is added with the partial products. These are given on the margin; the quotient being the multiplier; if the divisor were made multiplier the terms would be the same, but the work would occupy only two lines: by both operations, these several terms, together with the remainder, are seen to make up the given product or dividend, which is in this case incapable of exact analysis into partial products.

$$
\begin{array}{r}
x^4+ax^3 \\
-ax^3-a^2x^2 \\
a^2x^2+a^3x \\
-a^3x-a^4 \\
\hline
x^4+a^4 \\
+2a^4 \\
\hline
x^4-ax^3+a^2x^2-a^3x \\
ax^3-a^2x^2+a^3x-a^4 \\
\hline
x^4+a^4
\end{array}
$$

The chance of error in this process, from neglecting to change the signs of the partial products before adding them, may be considerably lessened, by setting down each term of the quotient, as it occurs, with the contrary sign to that which it ought to have, as, in multiplying, the

in finding the partial products is more readily performed, as the factors are brought close together; it is the mode of placing the quantities, both in Arithmetic and Algebra, adopted by French authors: the only English work in which, so far as we know, it is introduced, is the excellent treatise on Algebra by Professor Thomson of Glasgow.

signs of the partial products will be thus at once changed, and the subtractions converted into additions.*

The work will then be as follows; at the conclusion of the operation, the signs of the quotient must be again changed, to obtain the correct answer.

$$
\begin{array}{l|l}
x^4+a^4 & x+a \\
\hline
-x^4-ax^3 & -x^3+ax^2-a^2x+a^3 \quad \text{quotient with contrary signs} \\
\hline
\quad -ax^3+a^4 & \quad x^3-ax^2+a^2x-a^3 \quad \text{correct quotient} \\
\quad +ax^3+a^2x^2 & \\
\hline
\qquad a^2x^2+a^4 & \\
\qquad -a^2x^2-a^3x & \\
\hline
\qquad\quad -a^3x+a^4 & \\
\qquad\quad +a^3x+a^4 & \\
\hline
\qquad\qquad 2a^4 &
\end{array}
$$

3. Divide $6a^3x-17a^4x^2+30a^5x^3-25a^6x^4+12a^7x^5$ by $2a^2x-3a^3x^2+4a^4x^3$.

$$
\begin{array}{l|l}
6a^3x-17a^4x^2+30a^5x^3-25a^6x^4+12a^7x^5 & 2a^2x-3a^3x^2+4a^4x^3 \\
\hline
6a^3x-\ 9a^4x^2+12a^5x^3 & 3a-4a^2x+3a^3x^2 \\
\hline
\quad -8a^4x^2+18a^5x^3-25a^6x^4 & \\
\quad -8a^4x^2+12a^5x^3-16a^6x^4 & \\
\hline
\qquad\quad 6a^5x^3-\ 9a^6x^4+12a^7x^5 & \\
\qquad\quad 6a^5x^3-\ 9a^6x^4+12a^7x^5 & \\
\hline
\qquad\qquad 0 &
\end{array}
$$

Let this Ex. be also worked by the method explained in Ex. 2.

4. Divide $x^4-4x^3-34x^2+76x+105$ by $x-7$.

This question, as well as all others belonging to this third case, may be briefly worked as follows:—The first

* The same object will be attained by changing the signs of the terms of the divisor. If this be done, it must be remembered, that in performing the successive divisions by the first term of the divisor, it must be taken with its original sign, as otherwise the signs of the quotient would be reversed.

term only of the remainders being required in order to determine the successive terms of the quotient, we may, to find these, perform mentally the additions into which the subtractions are converted by the method used in Ex. 2, and set down the partial products only.

$$
\begin{array}{l|l}
x^4-4x^3-34x^2+76x+105 & (x-7 \\
\quad -x^4+7x^3 & \quad -x^3-3x^2+13x+15. \\
\quad\quad -3x^3+21x^2 & \quad\quad \text{that is} \\
\quad\quad\quad +13x^2-91x & \quad\quad x^3+3x^2-13x-15. \\
\quad\quad\quad\quad +15x-105 & \\
\hline
\quad\quad\quad\quad\quad 0 &
\end{array}
$$

First terms of remainders, $3x^3$, $-13x^2$, $-15x$.

Here, the first term of the first remainder is $3x^3$; then $3x^3 \div -x = 3x^2$; putting this in the quotient with its sign changed, and multiplying we have the second partial product, $-3x^3+21x^2$. Adding now $21x^2$ to $-34x^2$, and dividing mentally, we have $13x$ for the next term of the quotient, and $13x^2-91x$ for the third product. Again, performing mentally the addition of the terms containing x, and then the division, and multiplying, we find $15x-105$ for the fourth product. The sum of these partial products, and the given product, or dividend, is of course zero. The work is thus seen to occupy half the space that it would take up by the ordinary method. To aid the memory, the first terms of the sums *(remainders* by the uncontracted process) might be set down thus, ($3x^3$, $-13x^2$, $-15x$), in the blank space to the right or left of the partial products, or in any other place, apart.

5. Divide $1+ax+bx^2+cx^3+dx^4$ &c. by $1-x$.

$$1-x)\,1+ax+bx^2+cx^3+dx^4\ \&c.\ \Big(1+(1+a)x+(1+a+b)x^2 +(1+a+b+c)x^3\ \&c,$$

$$\begin{array}{l} 1-x \\ \hline (1+a)x+bx^2 \\ (1+a)x-(1+a)x^2 \\ \hline (1+a+b)x^2+cx^3 \\ (1+a+b)x^2-(1+a+b)x^3 \\ \hline (1+a+b+c)x^3+dx^4 \\ (1+a+b+c)x^3-(1+a+b+c)x^4 \\ \hline (1+a+b+c+d)x^4+ex^5,\ \&c. \end{array}$$

This quotient might be arranged as under:

$$\begin{array}{c|c|c|c|} 1+1 & x+1 & x^2+1 & x^3+1 & x^4+,\ \&c. \\ a & a & a & a & \\ & b & b & b & \\ & & c & c & \\ & & & d & \end{array}$$

45. When the given quantities are such, that the literal part of the terms of the quotient can be supplied by inspection, (Art. 39) the method of "detached coefficients" may be advantageously employed in division. Thus, using the contracted mode of division given in Ex. 4,* the work of Ex. 3 will be as follows;

$$\begin{array}{rrrrr|rrr} 6 & -17 & +30 & -25 & +12 & 2 & -3 & +4 \\ -6 & +9 & -12 & & & -3 & +4 & -3 \\ & +8 & -12 & +16 & & 3 & -4 & +3 \\ & & -6 & +9 & -12 & & & \\ \hline & & & 0 & & & & \end{array}$$

[signs of quotient with contrary signs; correct quotient]

Remainders, -8, $+6$.

* The ordinary uncontracted process, may however be employed. Ex. 3 worked in this way will be as under,

$$\begin{array}{rrrrr|rrr} 6 & -17 & +30 & -25 & +12 & 2 & -3 & +4 \\ 6 & -9 & +12 & & & 3 & -4 & +3 \\ \hline & -8 & +18 & -25 & & & & \\ & -8 & +12 & -16 & & & & \\ \hline & & 6 & -9 & +12 & & & \\ & & 6 & -9 & +12 & & & \\ \hline & & & 0 & & & & \end{array}$$

Here the process is conducted as already explained in Ex. 4; the literal part of the first term being a, and the order of the terms known, the literal part of the quotient is easily supplied. The first terms of the remainders, -8, and $+6$, are set down in a separate place.

1. Divide $6a^5x^3-19a^6x^5+48a^7x^7-44a^8x^9+33a^9x^{11}+20a^{10}x^{13}$ by $2a^2x-3a^3x^3+5a^4x^5$.

$$
\begin{array}{rrrrrr|l}
6 & -19 & +48 & -44 & +33 & +20 & 2\ -3\ +5 \\
\cline{7-7}
-6 & +9 & -15 & & & & -3\ +5\ -9\ -4 \\
 & +10 & -15 & +25 & & & 3\ -5\ +9\ +4 \\
 & & -18 & +27 & -45 & & \\
 & & & -8 & +12 & -20 & \\
\cline{4-6}
 & & & & 0 & &
\end{array}
$$

Hence the literal part of the first term being a^3x^2, the complete quotient is $3a^3x^2-5a^4x^4+9a^5x^6+4a^6x^8$.

2. Divide $15a^9+a^8b+3a^6b^3+6a^5b^4+4a^4b^5+11a^3b^6+13a^2b^7-10ab^8-b^9$ by $3a^3+2a^2b-2b^3$.

Here, putting zero for the wanting term both in the dividend and divisor, the work will proceed as underneath.

$$
\begin{array}{rrrrrrrrrr|l}
15 & +1 & \pm0 & +3 & +6 & +4 & +11 & +13 & -10 & -1 & 3\ +2\ \pm0\ -2 \\
\cline{10-10}
-15 & -10 & \pm0 & +10 & & & & & & & -5\ +3\ -2\ -3 \\
 & +9 & +6 & \pm0 & -6 & & & & & & +2\ -4\ -3 \\
 & & -6 & -4 & \pm0 & +4 & & & & & \text{or} \\
 & & & -9 & -6 & \pm0 & +6 & & & & 5\ -3\ +2\ +3 \\
 & & & & +6 & +4 & \pm0 & -4 & & & -2\ +4\ +3 \\
 & & & & & -12 & -8 & \pm0 & +8 & & \\
 & & & & & & -9 & -6 & \pm0 & +6 & \\
\cline{1-10}
\pm0 & \pm0 & \pm0 & \pm0 & \pm0 & \pm0 & \pm0 & +3 & -2 & +5 &
\end{array}
$$

Now as the literal part of the first term of the quotient is a^6, there will be in all seven terms, six containing the powers of a, and one independent of a; the three last

coefficients are therefore those of a remainder. In fact, the process must terminate at the term whose coefficient is 3, as it contains a^2, a power of a lower than that in the first term of the divisor. The complete quotient is therefore

$$5a^6-3a^5b+2a^4b^2+3a^3b^3-2a^2b^4+4ab^5+3b^6,$$

and the remainder, $3a^2b^7-2ab^8+5b^9$.*

EXERCISES.

1. Divide bc by c.
2. ..... x^3 by x.
3. ..... $-x$ by itself.
4. ..... $20x^3y$ by $5xy$.
5. ..... $-x$ by x.
6. ..... $15ax$ by $3x$.
7. ..... $12b^5$ by $4b^4$.
8. ..... $15a^3b^2c$ by $5a^2b$.
9. ..... $\sqrt{x}$ by $\sqrt{x}$.
10. ..... a^6 by a^5.
11. ..... $8x$ by $2x$.
12. ..... b^{2n} by b^n.
13. ..... a^n by a.
14. ..... a^3n by $-a^3n$.
15. ..... $24a^2b^3$ by $8a^2b^2$.
16. ..... $12abcdef$ by $4acf$.
17. ..... x^a by x^b.

* The abbreviated method of division exemplified in the text, is taken, with but slight changes, from the excellent "Course of Mathematics" by Professor S. H. Christie, of Woolwich, recently published. It is very neat and simple, and renders division by detached coefficients much more easily performed than by Horner's method, upon which it is an improvement for many purposes, as in finding the greatest common measure, one of the most important applications of division.

18. Divide $72ab^5x^2yz$ by $9b^5x^2z$.
19. ..... $-a$ by -1.
20. ..... $-30a^nx^my^5$ by $-6x^my^5$.
21. ..... -1 by -1.
22. ..... 1 by -1.
23. ..... $5a^{n+1}$ by $5a^{n-1}$.
24. ..... 2 by $\sqrt{2}$.
25. ..... a^{n-1} by a^{n+1}.
26. ..... $12a^nx^5y^2$ by $4a^2x^3y$.
27. ..... $8a$ by $\sqrt{8a}$.
28. ..... twice the cube of a by half its square.
29. ..... $-15b^5x^3y$ by $3b^2x^3$.
30. ..... $12ax^2$ by $4bx$.
31. ..... $4b$ by $3b$.
32. ..... x by $4x$.
33. ..... nm by np.
34. ..... x^n by x^{n-1}.
35. ..... x^{n-1} by x^{2n}.
36. ..... abc by ad.
37. ..... $8x$ by $4\sqrt{x}$
38. ..... b^4 by a^3.
39. ..... $-16a$ by $-8b$.
40. ..... $12a$ by $-4b$.
41. ..... a^p by a^q.
42. ..... a^3 by x^2.
43. ..... $3b\sqrt{y}$ by $2a\sqrt{y}$.
44. ..... b by c.
45. ..... $b^3xy^3p^2q^3$ by $5p^2x$.
46. ..... $8mpx$ by $2mxy$.
47. ..... $36z^{10}b^9x^5y^2$ by $9a^3b^5x^3y$
48. ..... $-3a^3b$ by $-3a^3b$.
49. ..... -1 by a.
50. ..... $-a$ by $-b$.

51. Divide $12b^2y+4by^2-16b$ by $4b$.

52. ..... $3abc+12abx-9a^2b$ by $3ab$.

53. ..... $-9ab^5x^4y+12b^2x^5y^2+15a^5b^4x^5y-3b^2a^2x^4y^5$ by $-3b^2x^5y$.

54. ..... $ab+b^2$ by $2b$.

55. ..... $a^2-2a+7-3x-1$ by -1.

56. ..... $by^n+by^{n+1}+by^{n+2}+by^{n+3}+$ &c. by y^n.

57. ..... $-3a+7x-10y^2+2z-1$ by -1.

58. ..... $7n^2x^3-8xy^3-12bxy^2+13mnx$ by $4nx$.

59. ..... $5a^2x^3y^4-10x^5y^4-20a^5x^ny^3$ by $5x^3y^3$.*

60. ..... $a^2-2ab+b^2$ by $a-b$.

61. ..... a^2-x^2 by $a-x$.

62. ..... $x^2+3x-28$ by $x-4$.

63. ..... $x^2+2xy+y^2$ by $x+y$.

64. ..... m^2-36 by $m+6$.

65. ..... $64-x^2$ by $8-x$.

66. ..... $n^2+2np+p^2$ by $n+p$.

67. ..... $a-1$ by $\sqrt{a}+1$.

68. ..... $x^3-9x^2+27x-27$ by $x-3$.

69. ..... $a^{n+1}+ab^n-a^nb-b^{n+1}$ by a^n+b^n.

70. ..... $a^{n+1}+2a^nx+2a^{n-1}x^2+2a^{n-2}x^3$, &c. by $a+x$.

71. ..... $1-5x+10x^2-10x^3+5x^4-x^5$ by $1-2x+x^2$.

72. ..... $4x^5+4x^2-29x+21$ by $2x-3$.

* Such examples as the following come under the Second Case, though the divisor has a compound form; for every term is divisible by the compound factor:

$$3b(a+x)\overline{)\dfrac{3ab(a+x)^3-6b(a+x)^2+3b(a+x)}{a(a+x)^2-2(a+x)+1}}$$

$$a+b\overline{)\dfrac{6a(a+b)^3-8(a+b)^2-3x(a+b)}{6a(a+b)^2-8(a+b)-3x}}$$

73. Divide $x-100$ by $\sqrt{x}-10$.

74. ..... $1-n^2$ by $1-n$.

75. ..... x^4-a^4 by $x-a$.

Ans. $x^3+ax^2+a^2x+a^3$.

76. ..... x^4-a^4 by $x+a$.

Ans. $x^3-ax^2+a^2x-a^3$.

77. Divide $x^{pq}-1$ by x^q-1.

Ans. $x^{q(p-1)}+x^{q(p-2)}+x^{q(p-3)}\ldots+x^{2q}+x^q+1$.

78. ..... x^4+a^4 by $x-a$.

Ans. $x^3+ax^2+a^2x+a^3+\frac{2a}{x-a}$.

79. ..... $6x^4-96$ by $3x-6$.

Ans. $2x^3+4x^2+8x+16$.

80. ..... a^6-x^6 by $a-x$.

Ans. $a^5+a^4x+a^3x^2+a^2x^3+ax^4+x^5$.

81. ..... n^5-p^5 by $n-p$.

Ans. $n^4+n^3p+n^2p^2+np^3+p^4$.

82. ..... n^5+p^5 by $n+p$.

Ans. $n^4-n^3p+n^2p^2-np^3+p^4$.

83. Divide x^9+a^9 by $x+a$.

Ans. $x^8-ax^7+a^2x^6-a^3x^5+a^4x^4-a^5x^3+a^6x^2-a^7x+a^8$.

84. Divide x^6-1 by $x-1$.

Ans. $x^5+x^4+x^3+x^2+x+1$.

85. Divide a^n+b^n by $a+b$.

Ans. $a^{n-1}-a^{n-2}b+a^{n-3}b^2-a^{n-4}b^3$, &c.

86. Divide $12a^2+26ab-36ac+18ad-10b^2+29bc-6bd-21c^2+9cd$ by $6a-2b+3c$. *Ans.* $2a+5b-7c+3d$.

87. Divide $8a^5x^5+6a^6x^4-15a^7x^3+a^9x$ by $4a^2x^3-3a^3x^2-a^4x$. *Ans.* $2a^3x^2+3a^4x-a^5$.

88. Divide $6x^5-x^4y-13x^3y^2+10x^2y^3-2xy^4$ by $2x^3-3x^2y+xy^2$. *Ans.* $3x^2+4xy-2y^2$.

89. Divide $4ax^3+51a^2x^2+10a^4-48a^3x-15x^4$ by $4ax-5a^2+3x^2$ *Ans.* $-2a^2+8ax-5x^2$.

90. Divide $21x^3y^2+25x^2y^3+68xy^4-40y^5-56x^5-18x^4y$ by $5y^2-8x^2-6xy$. *Ans.* $-8y^3+4xy^2-3x^2y+7x^3$.

91. Divide $x-5x^3+7x^5-5x^7-6x^9+28x^{11}$ by $1-2x^2-4x^4$. *Ans.* $x-3x^3+5x^5-7x^7$.

92. Divide $12a^5b-b^6+2ab^5+8a^4b^2-9a^2b^4-13a^3b^3$ by $3a^2+2ab-b^2$. *Ans.* $4a^3b-3ab^3-b^4+\frac{ab^5-2b^6}{3a^2+2ab-b^2}$.

93. Divide $a^9-4a^7x^2+5a^6x^3-5x^9+14ax^8-20a^2x^7-10a^4x^5+19a^3x^6$ by $a^4+5x^4+2a^3x-4ax^3-3a^2x^2$. *Ans.* $a^5-2a^4x+3a^3x^2-3a^2x^3+2ax^4-x^5$.*

MISCELLANEOUS EXERCISES.

1. What is the sum of the sum and difference of two quantities?

2. What is the difference of the sum and difference of two quantities?

3. If the less of two quantities is x, and the difference 12, what is the greater?

4. If y be the greater of two quantities and d the difference, what is the less?

5. What is the difference between the square of the sum, and the sum of the squares, of two quantities?

6. Find the difference between the square of the sum and the square of the difference of two quantities?

7. If p be contained q times in a with a remainder r, what is the expression for a?

* From a comparison of Exercises, 75 to 85, and Example 2, it will appear that the difference of two equal powers is always divisible by the difference of the roots; that the difference of two equal powers is divisible by the sum of the roots, when the index is an even number; and the sum of two equal powers divisible by the sum of the roots, when the index is odd; but that in other cases there will be a remainder, after the division.

8. What does p^2+q^2 want of being a complete square?

9. What must be added to x^2+12x to make a complete square?

10. x persons have £a to pay, what is the share of each?

11. Express the two numbers whose digits are x and y, and x, y and z.

12. What are the factors of a^2-b^2, of $4n^2-64$, and of $81a^4-100x^2$?

13. A person paid a shillings for x articles, what is the price of each?

14. Find the squares of $6a-4x$, $2b-5y^3$ and $\frac{1}{2}a-8$, without actual multiplication.

15. What does $9-6x$ want of being a complete square?

16. If $s=\frac{1}{2}(a+b+c)$, what will $2s-2a$ be; and also $2s-2b$, and $2s-2c$?

17. If a man can do a piece of work in 12 days, how much of it will he do in 5, in 7, and in x days?

18. What are the factors of $a^2-(b-c)^2$, and of $(b+c)^2-a^2$?

19. The sum of two numbers is 56 and their difference 34, what are the numbers?

20. The half sum and half difference of two numbers are 18 and 12; find the numbers.

21. What will $4ax^3$ become when $x+h$ is put for x?

22. What does $(a+x)$ $(a-x)$ become when $a+b$ is substituted for x?

23. If $a-b$ be substituted for x in the expression $ax-bx$, what will that expression become?

24. If $x+h$ be substituted for x, in the expression x^4, what will the first term of the difference between x^4 and $(x+h)^4$ be?

CHAP. III.—FRACTIONS.*

ART. 46. DEF. I. When a less quantity is contained a certain number of times exactly in a greater, the greater is called a *multiple* of the less, and the less is said to be a *part, measure,* or *sub-multiple* of the greater. Thus, b is a measure or sub-multiple of ab, and ab a multiple of a, and also of b: also, 36 is a multiple of 9, and 9 a part, measure, or sub-multiple of 36; but 9 is not a measure of 40.

47. II. When a quantity is a measure of several others, it is called their *common measure:* thus, 9 is a common measure of 63, 54, and 72, as it is contained a certain number of times exactly in each of them; also $a+x$ is a common measure of a^2-x^2 and $a^2+2ax+x^2$.

48. III. When a quantity is a multiple of several others, it is called their *common multiple:* thus, 56 is a common multiple of 4, 7, 8, 14, and 28; also $12a^3b^2xy$ is a common multiple of $6a^2x$, $4b$, $3ax$, and $2by$.

49. IV. An *even* number is one which can be divided into two equal integral parts; and an *odd* number, one, which cannot be so divided.

* In Algebra, the term *fraction* is not applied, as in Arithmetic, to denote merely the quotient of a *less* quantity by a *greater*, but the division of any quantity by another, whether greater than it or not, when the division is *indicated* only, *not performed*; the dividend and divisor are called the numerator and denominator, as in Arithmetic, and when spoken of together, they are called the *terms* of the fraction.

50. *Corollary.* Hence, if n represent any number whatever, even or odd, $2n$ will be always even, as it is divisible by 2. Hence, also, a number greater or less than $2n$ by unity, will be odd: thus, if $n=7$, $2n+1=15$, and $2n-1=13$, both odd; if $n=18$, $2n\pm1=37$ or 35, both odd, &c.; so that all odd numbers may be represented by the expression $2n\pm1$. This is generally expressed by saying that even numbers are of *the form* $2n$, and odd numbers of *the form* $2n\pm1$. Even and odd numbers may, however, be represented by other expressions besides these; for, if n and n' be any two numbers, $2(n\pm n')$ will be even; but $2(n\pm n')\pm1$ will odd. Thus, if $n=9$, and $n'=6$, $2(n\pm n')=30$ or 6, even, and $2(n\pm n')\pm1=31$ or 29, 7 or 5, all odd. So also $2(2nn')$, $2(n\pm n'\pm1)$, $2(nn'\pm n\pm n')$, are even forms, or of the form $2n$; and $2(n\pm n'\pm1)\pm1$, $2(2nn'\pm n)\pm1$, &c. are odd forms, or of the form $2n\pm1$.

51. V. A *prime* number is one which is not composed of any integral factors; and a *composite* number, one which is composed of two or more such factors.

52. *Cor.* Hence, a prime number cannot be divided by any numbers but itself and unity. Hence, also, no even number but 2 is prime.*

* Those factors of composite numbers which are themselves composite, can always be resolved into the same prime numbers. Thus, the factors of 30 are 5 and 6, and 3 and 10; now $6=3\times2$, therefore $30=5\times3\times2$, the factors being all prime. Also, $10=5\times2$, therefore $30=3\times5\times2$, the same prime factors as before. In like manner, the factors of 24 are 4 and 6, 3 and 8, 2 and 12, but all these can be resolved into the same prime factors, 2, 2, 2, 3. Thus, also, $360=2\times180$, $180=2\times90$, $90=2\times45$, $45=3\times15$, $15=3\times5$, therefore $360=2\times2\times2\times3\times3\times5$; and whatever composite factors of 360 are taken, they can be always resolved into these same prime numbers.

53. VI. *Commensurable* quantities are those which have a common measure greater than unity; and *incommensurable* quantities, or quantities *prime to each other*, are those which have no common measure greater than unity. Thus, 6 and 8, $4ax$ and $5ax^2$, are commensurable; while 5 and 6, 7 and 2, ax and by, $\sqrt{3}$ and $\sqrt{8}$, 2 and $\sqrt{5}$, are incommensurable, or prime to one another.

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54. LEMMA I. If a quantity measure another, it will measure any multiple of it.

For if $a=bc$, then $ma=mbc$; or if b be contained c times in a, it will be contained mc times in ma.

55. II. If a quantity measure each of two others, it will also measure their sum and difference, or the sum and differenc of any multiples of them.

Let p divide a and b, and let the quotients be q and q' then $\frac{a}{p}=q$, $\frac{b}{p}=q'$; therefore, by the nature of division, $a=pq$ and $b=pq'$, whence (Art. 27, 28,) $a\pm b=pq\pm pq'$, or $a\pm b=p(q\pm q')$; or (Art. 30), dividing the equals by p, $\frac{a\pm b}{p}=q\pm q'$. Now, since by hypothesis a and b are divisible by p, and q and q' are the quotients, q and q' are whole numbers, therefore their sum and difference are also whole numbers; that is, $q\pm q'$ is an integer, and consequently $a\pm b$ is divisible by p.

Again, let ma and nb represent any multiples of a and b; then, since $a=pq$, and $b=pq'$, $ma=mpq$, and $nb=npq'$: hence

$$ma\pm nb=(mq\pm nq')p,$$

and

$$\frac{ma\pm nb}{p}=mq\pm nq';$$

but m, n, q, *and* q', being integers, the quotient $mq \pm nq'$ is also an integer, and $ma \pm nb$ is therefore divisible by p.*

56. *Cor.* It follows, from this proposition, that if a number divide the whole of another number and a part of it, it will also divide the other part; and that if a number consist of several parts, each of which has a common divisor, p, then will the whole number composed of these parts be divisible by p.

57. III. The greatest common measure of two quantities, A and B, is the greatest common measure of Aa and Bb, a and b being integer multipliers, prime to one another, and such that there is no measure common to a and B, or to b and A.

For if d be the greatest common measure of A and B, so that $\frac{A}{d}=p$ and $\frac{B}{d}=q$, or $A=pd$ and $B=qd$, p and q can have no common measure. We have similarly $\frac{Aa}{d}=ap$ and $\frac{Bb}{d}=bq$ or $Aa=apd$ and $Bb=bqd$, and ap and bq can have no common measure; no new factor is introduced, and those common to A and B are common also to Aa and Bb, and the same is obviously true of A and Bb, or Aa and B, for this is only supposing either of the multipliers to be unity. Again, conversely, on the same supposition, the G. C. M. of Aa and Bb is also the G. C. M. of A and B. For if $\frac{Aa}{d}=ap$ and $\frac{Bb}{d}=bq$, then $\frac{A}{d}=p$ and $\frac{B}{d}=q$, and since

* Thus, if a and b be 54 and 42, then 6, which is a divisor of both, divides also 54 ± 42, or 96 and 12, the quotients being 16 and 2, both integers. And if m and n be 4 and 5, we have $\frac{216 \pm 210}{6}=36 \pm 35$, or $\frac{426}{6}=71$, and $\frac{6}{6}=1$, integers as before.

ap and bq have no common measure, it follows, that p and q also have no common measure; and the G. C. M. of Aa and Bb is also the G. C. M. of A and B.*

58. PROB. I. *To find the greatest common measure of two algebraic quantities. First*, If the divisor sought be *simple*, to the greatest common measure of the numerical coefficients annex the letters common to all the quantities.

Thus, the greatest common measure of $24ab^4n^2y^5$ and $56an^2x^2y^3$ is $8an^2y^3$.

Secondly, If the divisor sought be *compound*, the following rule must be employed:

(1.) Divide either quantity, or both, by the greatest simple factor contained in it, (Art. 57); (2) and having arranged the terms as in Division (Art. 44.), divide the one of higher by the one of lower dimensions, if their degrees be different; (3) if there be a remainder, divide the last divisor by it; (4) continue the process till there is no remainder, dividing always the preceding divisor by the preceding remainder, and at

* Thus, if we take the numbers 30 and 36, whose G. C. M. is 6; and if we multiply them respectively by 7 and 11, the products $6\times5\times7$ and $6\times6\times11$, or 210 and 396 have the same common measure, 6, as before, the multipliers being prime to one another, and to the given numbers. Also, if we multiply 30 by 5, and 36 by 9 the products $6\times5\times5$ and $6\times6\times9$, have still 6 as their G. C. M.; whereas if we multiply 30 by 9 and 36 by 5, the products $6\times5\times9$ and $6\times6\times5$, or 270 and 180 have $6\times5\times3$ or 90 for their G. C. M., a factor of each number being introduced into the multiple of the other, which is contrary to the hypothesis of a being prime to B and b to A. Further, if one only of the given numbers, as 30, be multiplied by any number, as 5, which has no factor in common with 36, the G. C. M. of 30, and the product 150, is not altered.

the same time multiplying or dividing the dividend, and the successive divisors and dividends by any quantity which has no factor in common with the original divisor, or dividend (Art. 57); the last divisor will be the common measure required.

The following is the general form of the process: the multipliers $a, b, c, \ldots$ fulfilling the conditions of Lem. III. and thus rendering the quotients $p, q, \ldots$ integral in every case, fractional results being excluded by the nature of the operation. (Art. 54, 55, 57.)

Here, B being supposed to represent the quantity of lower dimensions, and a a factor introduced to avoid fractions, B is contained p times in Aa with a remainder Cc; where C is a compound quantity, and c a quantity, either simple or compound, and not a factor of both A and B; take C alone for the new divisor, and for the new dividend the last divisor B or a multiple, Bb, of it, if necessary; and so on till there is no remainder. Then the last divisor, D, is the greatest common measure of A and B. *

$$
\begin{array}{l}
B)A \\
B)Aa\ (p \\
\quad \underline{pB} \\
\quad Cc)\ B \\
\qquad C)\ Bb\ (q \\
\qquad\quad \underline{qC} \\
\qquad\quad Dd)\ C \\
\qquad\qquad D)\ C\ (r \\
\qquad\qquad\quad \underline{rD} \\
\qquad\qquad\quad \ldots
\end{array}
$$

To prove this it is necessary to shew that,

1.—Every measure of A and B is a measure of D.

* That the above process must at last terminate, may be proved as follows:—Rejecting the multipliers a and c for the sake of simplicity, we shall have $A = pB + C$. Now, $B > C$ and therefore $pB > C$, hence $pB + C > 2C$, that is, $A > 2C$, and $\frac{1}{2}A > C$, or $C < \frac{1}{2}A$; therefore $pB > \frac{1}{2}A$, and from A a quantity greater than its half has been taken. In the same manner it might be shown, that when C is made dividend more than its half is taken away, and so on with respect to every successive dividend, so that at last there will remain a quantity less than anything that can be assigned.

For whatever quantity, m, measures A and B, it will measure Aa and pB (Art. 54), and therefore Aa—pB (Art. 55), that is Cc; and since m measures Cc it measures also C, because, by hypothesis, c is prime to A and B, and therefore also to m, which is a measure of A and B. Now m, measuring B and C, will measure also Bb and qC, and therefore also their difference Dd; and since m measures Dd it will measure D, for d is prime to A and B and therefore to m, their measure. Thus any measure, m, of A and B is a measure also of D.

2.—D is a measure of A and B.

For D is by hypothesis a measure of C, being contained in it without remainder; it is evidently a measure of Dd; it therefore measures qC+Dd or Bb, and consequently B also, since d is prime to A and B, and therefore to D, the measures of A and B having been just proved to be those of D. Now D measuring pB and Cc measures Aa, their sum, and therefore A also, since a is prime to A and B, and therefore to D. But it has been already shown that D measures B; it therefore measures both A and B.

3.—D is the greatest common measure of A and B.

For since every common measure of A and B is a measure of D, and since no quantity greater than D can measure it, therefore every common measure of A and B is *not greater* than D; now D is a common measure of A and B, and therefore there is none greater than it; that is, D is the greatest common measure of A and B.*

* The Arithmetical process for finding the greatest common measure of two numbers is of the same form as that in the text, but much less complicated; a proof is subjoined.

Let a and b be the numbers whose common measure is required, and let a be greater than b; the operation will be conducted as follows: .

59. *Cor.* 1. It is plain that if the given quantities have a simple common measure, it may be rejected before the

Dividing a by b, let the quotient be p, with a remainder c; again, making this remainder the divisor, and the former divisor the dividend, let c be contained in b, q times, with a remainder, d; and repeating the process, let d be contained r times in c, with no remainder: then it is asserted that d is the greatest common measure sought.

$$\begin{array}{r} b)\ a\ (p \\ \underline{pb} \\ c)\ b\ (q \\ \underline{qc} \\ d)\ c\ (r \\ \underline{rd} \\ 0 \end{array}$$

First, By the nature of division, $pb+c=a$, $qc+d=b$, and $rd=c$; then, since d measures c by the units in r, it measures qc (Art. 54); and since it measures qc and d, it measures also $qc+d$ (Art. 55), that is, b: it measures b, and therefore pb; it measures pb and c, and therefore $pb+c$, that is, a: but it was before shown to be a measure of b, it is therefore a measure both of a and b.

Secondly, If a number measure a and b, it measures a and pb, and therefore their difference c (Art. 55); also it measures b and c, and therefore b and qc, and consequently their difference d: hence, *every number* which measures a and b measures d. It follows from this, that the *greatest number* which measures a and b, measures d. Now, no number greater than d can measure d, therefore no number greater than d can measure a and b; that is, d is the greatest common measure of a and b. Hence, to find the greatest common measure of two numbers, divide the greater by the less, and the last divisor by the last remainder, until there is no remainder: the last divisor is the measure required.

Cor. The greatest common measure of three numbers, a, b, c, is obtained by finding d the greatest common measure of a and b, and then the greatest common measure of d and c is the greatest common measure of a, b, c: because every common measure of a and b, is a measure of d, and therefore the greatest common measure of d and c is the greatest common measure of a, b, c.

On the same principle, the greatest common measure of four or more numbers may be found.

EXERCISES.

1. Find the G. C. M. of 112 and 360. *Ans.* 8.
2. Find the G. C. M. of 168 and 360. *Ans.* 24.
3. Find the G. C. M. of 87 and 25. *Ans.* 1.
4. Find the G. C. M. of 2145 and 3471. *Ans.* 39.
5. Find the G. C. M. of 888 and 2775. *Ans.* 111.
6. Find the G. C. M. of 325, 455 and 312. *Ans.* 13.
7. Find the G. C. M. of 1404, 2457 and 3276. *Ans.* 117.

operation is begun; and that it must be afterwards introduced as a factor into the compound measure.

60. *Cor.* 2. The greatest common measure of more than two quantities will be found by taking the G. C. M. of two of them, the G. C. M. of the result and a third, and so on whatever number there be. Thus, if A, B, C, D, be the quantities, find m, the G. C. M. of A and B, n, the G. C. M. of m and C, &c.; then because every common measure of A, B, and C measures m and C, and every measure of m and C measures A, B and C, therefore n, the G. C. M. of m and C, must be the G. C. M. of A, B, and C; and the same may be shown of the others.

EXAMPLES.

1. Required the G. C. M. of the quantities,

$$2b^3-10ab^2+8a^2b \text{ and } 9a^4-3ab^3+3a^2b^2-9a^3b.$$

The former of these is divisible by $2b$, the latter by $3a$, (1); this gives

$$b^2-5ab+4a^2 \text{ and } 3a^3-b^3+ab^2-3a^2b.$$

Arranging the terms according to the powers of a (2), multiplying to avoid fractions (4), and then dividing, we have

$$\begin{array}{r} 3a^3-3a^2b+ab^2-b^3 \\ \hline 4 \\ \hline 4a^2-5ab+b^2)\overline{12a^3-12a^2b+4ab^2-4b^3}(3a \\ \underline{12a^3-15a^2b+3ab^2} \\ 3a^2b+ab^2-4b^3 \end{array}$$

Dividing this remainder by b, and multiplying (4) the last divisor, which is the new dividend, by 3, which is not a factor of the other quantity, we find

$$\begin{array}{r} 3a^2+ab-4b^2)12a^2-15ab+3b^2(4 \\ \underline{12a^2+4ab-16b^2} \\ -19ab+19b^2 \end{array}$$

Divide this remainder by $-19b$, and make the last divisor the dividend; the work will then terminate:

$$a-b)3a^2+ab-4b^2(3a+4b$$
$$3a^2-3ab$$
$$4ab-4b^2$$
$$4ab-4b^2$$
$$0$$

Hence, $a-b$ is the G. C. M. required.

2. Required the G. C. M. of $36x^6-18x^5-27x^4+9x^3$ and $27x^5y^2-18x^4y^2-9x^3y^2$.

Here $9x^3$ is obviously a factor of both quantities, and therefore (Art. 59) of the G. C. M. Dividing by $9x^3$, rejecting y^2 which is a factor of the latter quantity only, and, to avoid fractions, multiplying the dividend by 3, which is not a factor of the divisor, there is obtained, after division, the remainder,

$$2x^2-5x+3=2x(x-1)-3(x-1)=(2x-3)\ (x-1).$$

Rejecting $2x-3$, which is plainly not a factor of both quantities, $x-1$ is contained in the preceding divisor, without remainder. Hence, $9x^3(x-1)$ is the G. C. M. required.

3. Required the G. C. M. of $14x^3+7x^2-56x+35$ and $7x^2-12x+5$.

Employing the method of division by detached coefficients, given in Art. 45, the work will stand thus:

14+ 7—56+35	7—12+ 5	—2—4
—14+24—10	—7+42—35	—7
—28+48—20		
3—18+15	30—30	
or 1—6+5	or 1— 1	—1+5
—1+1		
5—5		
±0±0±0		

Here placing the divisor to the right of the dividend,

the quotient, with contrary signs, is $-2-4$, and the remainder divided by 3, gives $1-6+5$ for a new divisor. Dividing by this the first divisor, $7-12+5$, we have -7 for quotient, and $30-30$, or, by division, $1-1$, for the next divisor; dividing $1-6+5$ by it, the quotient is $-1+5$, and there is no remainder. Hence $+1, -1$ are the coefficients of the G. C. M., it is therefore plainly $x-1$, since 12, 42, and 30, have x as their literal part.

4. Find the G. C. M. of the quantities $x^5-3ax^4+2a^2x^3+2a^3x^2-3a^4x+a^5$ and $x^5-ax^4+a^2x^3-a^3x^2-2a^4x+2a^5$.

$1-3+2+2-3+1$	$1-1+1-1-2+2$	-1
$-1+1-1+1+2-2$	2	
$-2+1+3-1-1$	$2-2+2-2-4+4$	$+1$
$+2-10+6+10-8$	$-2+1+3-1-1$	
$-9+9+9-9$	$-1+5-3-5+4$	-2
or $1-1-1+1$	$+1-1-1+1$	$+1-4$
	$-4+4+4-4$	
	$\pm0\pm0\pm0\pm0\pm0$	

Here, the terms being arranged as in last example, we have for the first remainder $-2+1+3-1-1$; the first divisor is doubled to render it divisible by this, subtracting we find $-1+5-3-5+4$, the first term of which is not divisible by 2; 4 is therefore annexed, and $-1+5-3-5+4$ made the new divisor. Dividing the preceding divisor by this, the next remainder is $-9+9+9-9$, which being divided by -9 gives $1-1-1+1$ for the next divisor; this is contained without remainder in the preceding divisor, $-1+5-3-5+4$, and the work terminates. Hence, $1-1-1+1$ represents the G. C. M., which by a comparison of the terms is easily seen to be $x^3-ax^2-a^2x+a^3$.

This example may also be worked by making the dividend divisor, and the divisor dividend, since both are of the same dimensions, and have unity for the coefficient of the first term; the work by this arrangement of the quantities will form perhaps a better exercise for the student than an additional example.

1—1+1—1—2+2	1—3+ 2+ 2— 3+1	—1
—1+3—2—2+3—1	2	
2—1—3+1+1	2—6+ 4+ 4— 6+2	—1+5
—2+2+2—2	—2+1+ 3— 1— 1	
—1+1+1—1	—5+ 7+ 3— 7+2	
	2	
±0±0±0±0±0	—10+14+ 6—14+4	
	10— 5—15+ 5+5	
	9— 9— 9+9	
	or, 1— 1— 1+1	—2—1

The divisor being placed to the right of the dividend as before, the first quotient is —1, and remainder 2—1—3+1+1; the first divisor is then doubled to render it divisible by this remainder; the division being then performed it is seen, by performing the addition mentally, that the first term of the remainder would be —5, and therefore not divisible by the first term, 2, of the divisor, the full remainder is therefore brought down and doubled, and the division proceeded with; the remainder divided by 9 gives 1—1—1+1 for the new divisor; by this the preceding divisor 2—1—3+1+1 is divided and there is no remainder. The G. C. M. then is, as before, $x^3—ax^2—a^2x+a^3$.

EXERCISES.

1. Find the G. C. M. of $x^3—a^3$ and $x^2—a^2$. *Ans.* $x—a$.

2. Find the G. C. M. of a^4-x^4 and $a^3+a^2x-ax^2-x^3$.

Ans. a^2-x^2.

3. Find the G. C. M. of $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$. *Ans.* $2a+3x$.

4. Find the G. C. M. of $15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4$ and $12a^3b^2+38a^2b^3+16ab^4-10b^5$. *Ans.* $3a^2+2ab-b^2$.

5. Find the G. C. M. of $3x-5-18x^2+4x^4+2x^3$ and $-11x^3-3x^2-1-3x+6x^5-4x^4$. *Ans.* $2x^3-4x^2+x-1$.

6. Find the G. C. M. of y^2+5y+4, y^2+2y-8, and $y^2+7y+12$. *Ans.* $y+4$.

7. Find the G. C. M. of $ax^2-a^2x+14x^3+a^3$ and $4ax^2-2a^2x+7x^3+a^3$. *Ans.* $7x^2-3ax+a^2$.

8. Find the G. C. M. of $12x^3+55x+63$ and $63x^3-36x^2-343x+196$. *Ans.* $3x+7$.

9. Find the G. C. M. of x^4+2x^2+9 and $7x^3-11x^2+15x+9$. *Ans.* x^2-2x+3.

10. Find the G. C. M. of $6x^6-x^5-13x^4+11x^3+5x^2-3x-1$, and $10x^5-19x^4+17x^3-6x^2+x+3$.

Ans. $2x^3-3x^2+x+1$.

11. Find the G. C. M of $x^6+4x^5-3x^4-9+12x+11x^2-16x^3$ and $6x^5-12x^3+12-48x^2+20x^4+22x$.

Ans. x^3+x^2-5x+3.

12. Find the G. C. M. of $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$. *Ans.* x^2-1.

61. *To find the least common multiple of two quantities.* Divide their product by their greatest common measure.

Let a and b be the quantities and x their greatest common measure; and let p and q be the quotients arising from dividing a and b by this measure, so that $\frac{a}{x}=p$, $\frac{b}{x}=q$; then, since x is the *greatest possible* divisor,

p and q have no common measure.* Hence, pq is the least common multiple of p and q, and therefore pqx is the least common multiple of px and qx. Now, $px=a$, and $qx=b$, therefore $ab=px\times qx$; whence, $\frac{ab}{x}=pqx$. Hence, $\frac{ab}{x}$, or $\frac{a}{x}\times b=pb$, is the least common multiple of a and b.

62. *Cor.* 1. It is plain, that the least common multiple of two incommensurable quantities is their product.

63. *Cor.* 2. The least common multiple of several quantities will be found by finding the least common multiple of the first two, the least common multiple of that multiple and the third, and so on.

64. *Cor.* 3. Since dividing quantities by their greatest common measures, is the same as detaching all the common factors, thus leaving results which are prime to one another, it is plain that the least common multiple of several quantities will be *the continued product of the common measures which exist between two or more of them, and the quotients which result from their division by all the common factors*. Thus, if the quantities be $4a$, $6ab$, $8abc$, we first detach the factor $2a$, the G. C. M., giving 2, $3b$, $4bc$, then the factor b, giving 2, 3, $4c$, and lastly the factor 2, giving 1, 3, $2c$; then $2a\times b\times 2\times 1\times 3\times 2c=24abc$, the least common multiple; the continued product of the quantities being $192a^3b^2c$. From this the reason of the common arithmetical process is manifest.

* For, if they had a common measure, it would enter as a factor into the divisor of a and b, and there would therefore be a measure greater than x; that is, we should have $\frac{a}{x}=np$, and $\frac{b}{x}=nq$; whence $\frac{a}{nx}=p$, and $\frac{b}{nx}=q$, which is contrary to the hypothesis.

EXAMPLES.

1. Required the L. C. M. of the quantities $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.

The G. C. M. of these quantities is $2a+3x$, (Exercise 3, Art. 60,) hence the quotient of the first quantity and this measure is $3a+x$; then $(6a^2+7ax-3x^2)\ (3a+x)=18a^3+27a^2x-2ax^2-3x^3$ is the L. C. M. required.

2. Required the L. C. M. of 325 and 455.

The G. C. M. of these numbers is 65; then $325\div 65=5$; therefore $455\times 5=2275$ is the L. C. M. required.

EXERCISES.

1. Find the L. C. M. of $8x^2$, $12x^3$, and $20x^4$.

Ans. $120x^4$.

2. Find the L. C. M. of $x-1$, $x+1$, x^2-1.

Ans. x^2-1.

3. Find the L. C. M. of $6ax$, $15bx$, $27b$, and $35abx$.

Ans. $1890abx$.

4. Find the L. C. M. of the nine digits. *Ans.* 2520.

5. Find the L. C. M. of x^3-a^3 and x^2-a^2.

Ans. $x^4+ax^3-a^3x-a^4$.

6. Find the L. C. M. of $x^3-3x^2+7x-21$ and x^4-49.

Ans. $x^5-3x^4-49x+147$.

7. Find the L. C. M. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

Ans. $a^5-a^4x-ax^4+x^5$.

65. PROP. I. A fraction is multiplied by any quantity when its numerator is multipled, or its denominator divided, by that quantity.

Let $\frac{a}{b}$ be the fraction, and x its value,* so that $\frac{a}{b}=x$;

* This is merely expressing the result of the division of a by b, by the simple symbol x, instead of the complex symbol $\frac{a}{b}$.

then $\frac{2a}{b} = \frac{a+a}{b}$; but, by the nature of division, $\frac{a+a}{b}$ $= \frac{a}{b} + \frac{a}{b} = x+x=2x$: also, $\frac{5a}{b} = \frac{a+a+a+a+a}{b} =$ $\frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \frac{a}{b} = x+x+x+x+x = 5x$; and in general, $\frac{na}{b} = \frac{a+a+a \text{ &c. to } n \text{ terms}}{b} = \frac{a}{b} + \frac{a}{b} + \frac{a}{b}$ &c. to n terms $= x+x+x$ &c. to n terms $= nx$, therefore $n \frac{a}{b}$ $= \frac{na}{b}$.

Again, let the fraction be $\frac{a}{bn}$, and x its value, so that $\frac{a}{bn} = x$; then, by the nature of division, $a = bnx$, and (Art. 30) $\frac{a}{b} = nx$; but $\frac{a}{b}$ is the form which the original fraction assumes, when its denominator is divided by n: therefore, the division of the denominator multiplies the original value of the fraction.

66. II. A fraction is divided by any quantity when its numerator is divided, or its denominator multiplied, by that quantity.

Let the fraction be $\frac{an}{b}$: this (Art. 65) is the product which arises from multiplying the fraction $\frac{a}{b}$ by n; if, therefore, this product be divided by one of the factors n,

the quotient will be the other factor $\frac{a}{b}$: hence, $\frac{an}{b} \div n = \frac{a}{b}$.

Again, let $\frac{a}{b}$ be the fraction, and x its value, so that $\frac{a}{b} = x$: if the denominator be multiplied by n, the fraction becomes $\frac{a}{nb}$; but to divide by a product is the same as to divide by its factors in succession, therefore $\frac{a}{nb} = (a \div b) \div n$: now, $a \div b = \frac{a}{b} = x$, therefore $(a \div b) \div n = \frac{x}{n}$, that is, $\frac{a}{nb} = \frac{x}{n}$; or the multiplication of the denominator divides the value of the fraction.

67. III. If the terms of a fraction be multiplied or divided by the same quantity, the value of the fraction is not changed.

Let $\frac{a}{b}$ be the fraction, and x its value, so that $\frac{a}{b} = x$; then, if the numerator be multiplied by n, the fraction will become $\frac{an}{b} = nx$ (Art. 65); and if the denominator of this new fraction be multiplied by n, we shall have $\frac{an}{bn} = x$ (Art. 66); but $x = \frac{a}{b}$, therefore $\frac{an}{bn} = \frac{a}{b}$.

Again, let $\frac{an}{bn}$ be the fraction, and x its value, so that $\frac{an}{bn} = x$: then, if the denominator be divided by n, the fraction will become $\frac{an}{b} = nx$ (Art. 65); and if the nume-

rator of this new fraction be divided by n, we get $\frac{a}{b} = x$ (Art. 66); but $x = \frac{an}{bn}$, therefore $\frac{a}{b} = \frac{an}{bn}$.

68. PROB. I. *To reduce a fraction to its lowest terms.* Find the greatest common measure of the terms (Art. 58), and divide both the terms by it.*

It follows from the nature of the greatest common measure, that when both terms of a fraction are divided by it, they are rendered prime to one another, and the fraction is therefore irreducible; the reduced fraction is of the same value as the original one by Art. 67.

EXAMPLES.

1. Reduce $\frac{24a^2x^3y^2}{8abx^4}$ to its lowest terms.

Here $8ax^3$ is plainly the highest divisor of both terms; dividing by this quantity, we find $\frac{3ay^2}{bx}$ for the fraction in its simplest form.

2. Reduce $\frac{a^2-x^2}{a^2+2ax+x^2}$ to its simplest form.

Here $a+x$ will obviously divide both terms: dividing, we have $\frac{a-x}{a+x}$

which is not capable of farther reduction.

3. Reduce the fraction $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

* Or, find by inspection any common divisor of the terms, and divide them both by it; the fraction is thus reduced to *lower* terms: repeat the process till no common divisor of the terms can be found, and the fraction will then be in its *lowest* terms.

Here, since 16 and 6 are the doubles, and 24 and 9 the triples of 8 and 3, it is plain that

$$\frac{2x^3-16x-6}{3x^3-24x-9} = \frac{2}{3}\left(\frac{x^3-8x-3}{x^3-8x-3}\right) = \frac{2}{3}.$$

4. Reduce $\frac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$ to its lowest terms.

The greatest common measure must, in this case, be found by Art. 58. The work is as follows:

$27+\ 63-12-\ 28$		
5		
$135+315-60-140$	$15+35+3+7$	-9
$-135-315-27-\ 63$	$-15-35$	$-5-1$
	$-3-7$	
$-87-203$	$\pm\ 0\pm 0\pm 0\pm 0$	
or, dividing by -29, $3+\ 7$		

Whence the last divisor, $3x+7$, is the greatest common measure; dividing both terms of the original fraction by it, we obtain, for the reduced fraction,

$$\frac{5x^2+1}{9x^3-4x}.$$

EXERCISES.

Fractions to be reduced.	Answers.
1. $\frac{ab}{cb^2}$	$\frac{a}{cb}$.
2. $\frac{9bde}{12fdb}$	$\frac{3e}{4f}$.
3. $\frac{2a^3x^2}{5a^2x^2b}$	$\frac{2a}{5b}$.
4. $\frac{a^3-b^3}{a^2-b^2}$	$\frac{a^2+ab+b^2}{a+b}$.
5. $\frac{a^3-b^3}{(a-b)^2}$	$\frac{a^2+ab+b^2}{a-b}$.

Fractions to be reduced.		Answers.
6.	$\frac{a^3x-2a^2x^2+ax^3}{a^4x+a^3x^2+a^2x^3+ax^4}$	$\frac{a^2-2ax+x^2}{a^3+a^2x+ax^2+x^3}$.
7.	$\frac{14a^2-7ab}{10ac-5bc}$	$\frac{7}{5}\cdot\frac{a}{c}$.
8.	$\frac{21a^3b^2x-9ab^3x^2}{15a^2b^2x+3a^5b^4x^2-12ab^3x}$	$\frac{7a^2-3bx}{5a+a^4b^2x-4}$.
9.	$\frac{n^3-2n^2}{n^2-4n+4}$	$\frac{n^2}{n-2}$.
10.	$-\frac{1-x}{1-x^2}$	$-\frac{1}{1+x}$.
11.	$\frac{5a^2+5ax}{a^2-x^2}$	$\frac{5a}{a-x}$.
12.	$\frac{x^2+2x-3}{x^2+5x+6}$	$\frac{x-1}{x+2}$.
13.	$\frac{x^3-39x+70}{x^2-3x-70}$	$\frac{x^2-7x+10}{x-10}$.
14.	$\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$	$\frac{3a^2+b^2}{4a-b}$.
15.	$\frac{10a^nb^3c^2-15a^{n+1}b^2c^3}{20a^{n+2}b^4c^2-5a^nb^2c^2}$	$\frac{2b-3ac}{4a^2b^2-1}$.
16.	$\frac{x^4-3x^3+x^2+3x-2}{4x^3-9x^2+2x+3}$	$\frac{x^3-2x^2-x+2}{4x^2-5x-3}$.
17.	$\frac{x^3-x^2-21x+45}{3x^2-2x-21}$	$\frac{x^2+2x-15}{3x+7}$.
18.	$\frac{a^n}{a^{n+1}}$	
19.	$\frac{n^2-9}{n^2-6n+9}$	
20.	$\frac{4a^2x^2y}{12a^3x^3b}$	
21.	$\frac{16bc^3x^2}{20ax^3c^2}$	
22.	$\frac{ax+x^2}{3bx-cx}$	

69. II. *To reduce an integer to the form of a fraction having a given denominator.* Multiply the integer by the given denominator, and under the product taken as a numerator set that denominator.

This is merely performing a process, and indicating its reverse; and therefore the value of the quantity is not changed.

EXAMPLES.

1. Reduce 5 to the form of a fraction with the denominator 7.

Here $5\times7=35$; and indicating the division by 7, we have

$$\frac{35}{7}$$

for the required form.

2. Express a by a fraction whose denominator shall be $b-x$.

Here $a\times(b-x)=ab-ax$; and putting $b-x$ for divisor, the fraction is

$$\frac{ab-ax}{b-x}, \text{ or } \frac{a\,(b-x)}{b-x}$$

3. Express $x-3$ by a fraction having the same denominator as $\frac{5a}{n+y}$.

Multiplying $x-3$ by $n+y$, and then indicating the division by $n+y$, we get for the required form

$$\frac{x\,(n+y)-3\,(n+y)}{n+y}, \text{ or } \frac{nx+xy-3n-3y}{n+y}$$

EXERCISES.

Quantities to be reduced.	ANSWERS.
1. Express a as a fraction with the denominator 1.	$\frac{a}{1}$.
2. ..... 3 with the denominator 7	$\frac{21}{7}$.

Quantities to be reduced.			ANSWERS.
3. Express x with the denominator b			$\frac{bx}{b}$.
4.	$a-x$	$a+x$	$\frac{a^2-x^2}{a+x}$.
5.	$12a^2x$	$5y^2$	$\frac{60a^2xy^2}{5y^2}$.
6.	11	9	$\frac{99}{9}$.
7.	$2a-1$	$a-1$	$\frac{2a^2-3a+1}{a-1}$.
8.	$x-2$	unity	$\frac{x-2}{1}$.
9.	$\sqrt{x}$	a	$\frac{a\sqrt{x}}{a}$.
10.	$6a$	of $\frac{3x}{7b}$	$\frac{42ab}{7b}$.
11.	$(a-x)^3$	of $\frac{b}{(a-x)^2}$	$\frac{(a-x)^5}{(a-x)^2}$

70. III. *To reduce a fraction to an integral or mixed quantity.* Divide the numerator by the denominator, and if there be a remainder, annex it as in division (Art. 44.)

EXAMPLES.

1. Express $\frac{4ax-6a^2}{2a}$ as an integral or mixed quantity.

Dividing the terms of the numerator by the denominator, we obtain in this case an integral quantity.

$2a)\overline{4ax-6a^2}$

$2x-3a$

2. Reduce $\frac{8b^2x-4ab^2-3a^2b}{4b^2}$ to an integer or to a mixed quantity.

Performing the actual division, we obtain $2x-a$, with a remainder of $-3a^2b$; this remainder, with the divisor subscribed, is then reduced.

$$4b^2)8b^2x-4ab^2-3a^2b$$

$$2x-a-\frac{3a^2b}{4b^2}$$

or

$$2x-a-\frac{3a^2}{4b}$$

EXERCISES.

Fractions to to be reduced.	ANSWERS.
1. $\frac{b(a^2-x^2)}{a-x}$	$b(a+x)$.
2. $\frac{1}{2c}(8c^2+6c+5)$	$4c+3+\frac{5}{2c}$.
3. $\frac{1}{3b}(6b^2x-9ab+21)$	$2bx-3a+\frac{7}{b}$.
4. $\frac{a(b^5+x^5)-c}{b+x}$	$ab^4-ab^3x+ab^2x^2-abx^3+ax^4-\frac{c}{b+x}$.
5. $\frac{1}{x-y}(x^3-y^3+x^2-2y^2)$	$x^2+xy+y^2+x+y-\frac{y^2}{x-y}$.
6. $\frac{1-b^2-a(1-b)}{1-b}$	$1+b-a$.
7. $\frac{4ax^2-6b^2x^2+2x^5-5b}{2x^2}$	$2a-3b^2+x-\frac{1}{2x^2}\cdot 5b$.

71. IV. *To reduce fractions having different denominators to equivalent ones having a common denominator.* Multiply both terms of each fraction by all the denominators except its own.

Since both terms of each fraction are multiplied by the same quantity, the new fractions are respectively equal (Art. 67) to the original ones: they must all have the same denominator, because each denominator is the product of the same factors.

EXAMPLES.

1. Reduce $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, to fractions having a common denominator.

Here, multiplying both terms of $\frac{a}{b}$ by d and f, $\frac{a}{b} = \frac{adf}{bdf}$

both terms of $\frac{c}{d}$ by b and f, and both terms of $\frac{e}{f}$ by b and d, we have the three results $\frac{c}{d} = \frac{cbf}{bdf}$ $\frac{e}{f} = \frac{ebd}{bdf}$ equal to the three original fractions respectively.

2. Reduce a, $\frac{3a-1}{4b}$, 6, to fractions having a common denominator.

Here, reducing a and 6 to fractions having the denominator $4b$ (Art. 69), we find for the three fractions

$$\frac{4ab}{4b}, \quad \frac{3a-1}{4b}, \quad \frac{24b}{4b}.$$

72. When the fractions are such that the denominators have a common multiple less than their continued product, the process of finding the common denominator may be much simplified, by applying the following rule;

Find the least common multiple of the denominators (Art. 61), and multiply both terms of each fraction by the quotient of the common multiple and the denominator of that fraction.

EXAMPLES.

1. Let it be required to reduce the fractions

$$\frac{a}{px}, \quad \frac{b}{qx}, \quad \frac{c}{ry}, \quad \frac{d}{sy},$$

to equivalent fractions having a common denominator.

Here the least common multiple of the denominators is $pqrsxy$, and this divided by the denominators in succession gives the quotients

$$qrsy,\ prsy,\ pqsx,\ pqrx;$$

whence, multiplying both terms by these quotients respectively, we obtain for the reduced fractions

$$\frac{aqrsy}{pqrsxy},\ \frac{bprsy}{pqrsxy},\ \frac{cpqsx}{pqrsxy},\ \frac{dpqrx}{pqrsxy}.$$

With respect to the reason of this, it is plain that, by using the least common multiple instead of the continued product, those factors are omitted which would be common to both terms of all the reduced fractions; and that the division of the common multiple by each denominator, and the multiplication of both terms by the quotient, will reproduce the common multiple each time for a denominator.

2. Reduce $\frac{5a}{4bd}, \frac{2c}{3b}, \frac{7x}{6d}$ to a common denominator.

Here the least common multiple is $12bd$; and $12bd$ divided by each of the denominators gives $3, 4d, 2b$: when both terms are multiplied by these, we find $\frac{15a}{12bd}, \frac{8cd}{12bd}, \frac{14bx}{12bd}$ for the reduced fractions. In most cases, however, we can at once discover by inspection the quantity by which both terms are to be multiplied.

EXERCISES.

Fractions to be reduced.			ANSWERS.	
1.	$\frac{a}{b}$,	$\frac{c}{d}$, ...	$\frac{ad}{bd}$,	$\frac{bc}{bd}$.
2.	$\frac{4a}{9bd}$,	$\frac{5c}{3b}$	$\frac{4a}{9bd}$,	$\frac{15cd}{9bd}$.

Fractions to be reduced.	ANSWERS.
3. $\frac{a}{bd}, \frac{c}{d}$	$\frac{a}{bd}, \frac{bc}{bd}$.
4. $\frac{2x+1}{b^2x}, \frac{3a-1}{x}$	$\frac{2x+1}{b^2x}, \frac{3ab^2-b^2}{b^2x}$.
5. $\frac{a}{6xy}, \frac{b}{3x}, \frac{c}{2y}$	$\frac{a}{6xy}, \frac{2by}{6xy}, \frac{3cx}{6xy}$.
6. $d, \frac{x+1}{3}, \frac{n}{m}$	$\frac{3dm}{3m}, \frac{mx+m}{3m}, \frac{3n}{3m}$.
7. $\frac{ax}{8bc}, \frac{3ay}{2b}$	$\frac{ax}{8bc}, \frac{12acy}{8bc}$.
8. $\frac{x}{a+b}, \frac{y}{a-b}, \frac{z}{a^2-b^2}$	$\frac{x(a-b)}{a^2-b^2}, \frac{y(a+b)}{a^2-b^2}, \frac{z}{a^2-b^2}$.
9. $\frac{6a-3}{5ax}, \frac{3y}{5a}$	$\frac{6a-3}{5ax}, \frac{3xy}{5ax}$.
10. $\frac{3}{5}, a, \frac{a-x}{a+x}$	$\frac{3(a+x)}{5(a+x)}, \frac{5a(a+x)}{5(a+x)}, \frac{5(a-x)}{5(a+x)}$.
11. $\frac{p-q}{a^2-n^2}, \frac{x}{a+n}$	$\frac{p-q}{a^2-n^2}, \frac{ax-nx}{a^2-n^2}$.
12. $\frac{a-x}{x}, \frac{a+x}{x(a^2-x^2)}, \frac{a-x}{a+x}, \frac{1}{a-x}$.	
Ans. $\frac{(a+x).(a-x)^2}{x(a^2-x^2)}, \frac{a+x}{x(a^2-x^2)}, \frac{x(a-x)^2}{x(a^2-x^2)}, \frac{x(a+x)}{x(a^2-x^2)}$.	

73. V. *To add fractions.* Reduce them, if necessary, to equivalent ones having a common denominator; add the numerators, and under their sum place the common denominator.

EXAMLPE.

Let it be required to add the fractions

$$\frac{a}{b} \text{ and } \frac{c}{d}$$

Reducing them to a common denominator (Art. 71), we find

$$\frac{ad}{bd} \text{ and } \frac{bc}{bd}$$

then

$$\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

Here ad and bc are both divided by the same divisor, bd; and since, by the nature of division, the sum of the quotients of any quantities by the same divisor, is equal to the quotient of the sum of those quantities by that divisor; and since also the reduced fractions are (Art. 67) equivalent to the original ones, it follows that

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

Hence fractions cannot be incorporated, unless they have the same denominator; they can only be combined by means of their signs.

EXERCISES.

Fractions to be added.	Answers.
1. $\frac{x}{b}+\frac{a}{b}$	$\frac{a+x}{b}$.
2. $\frac{a}{3}+\frac{a}{4}$	$\frac{7a}{12}$.
3. $\frac{a}{b}+\frac{3a}{4b}$	$\frac{7}{4}\cdot\frac{a}{b}$.
4. $\frac{m}{n}+\frac{n}{m}$	$\frac{m^2+n^2}{mn}$.
5. $\frac{1}{2}(a+b)+\frac{1}{2}(a-b)$	a.
6. $\frac{a+x}{a-x}+\frac{a-x}{a+x}$	$\frac{2(a^2+x^2)}{a^2-x^2}$.
7. $\frac{1}{1+x}+\frac{1}{1-x}$	$\frac{2}{1-x^2}$.

Fractions to be added.	Answers.
8. $\frac{b+x}{6a}+\frac{b-x}{3a}$	$\frac{3b-x}{6a}$.
9. $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x$	$1\frac{5}{12}x$.
10. $\frac{x}{1-x}+\frac{x^2}{(1-x)^2}$	$\frac{x}{(1-x)^2}$.
11. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$	$\frac{1}{abc}(bc+ac+ab)$.
12. $\frac{1}{3}(2a+1)+\frac{1}{5}(4a+2)+\frac{1}{7}a$	$\frac{169a+77}{105}$.
13. $\frac{5x^2+b}{3b}+\frac{4x^2+2b}{5b}$	$\frac{37x^2+11b}{15b}$.
14. $\frac{1}{4}(a-3x)+\frac{1}{5}(3a-5x)+\frac{1}{20}(3a-5x)$	$a-2x$.
15. $\frac{c}{x}+\frac{b}{x^3}+\frac{a}{x^5}$	$\frac{a+bx^2+cx^4}{x^5}$.
16. $\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}$	$\frac{100a+10b+c}{10^3}$.
17. $\frac{1}{3(1-x)}+\frac{2+x}{3(1+x+x^2)}$	$\frac{1}{1-x^3}$.
18. $\frac{1}{3(1+x)}+\frac{2-x}{3(1-x+x^2)}$	$\frac{1}{1-x^3}$.
19. $\frac{b}{d}+\frac{ad-bc}{d(c+dx)}$	$\frac{a+bx}{c+dx}$.
20. $\frac{3a}{2b}+\frac{c}{4b^2}$	
21. $\frac{a}{16}+\frac{b}{4}$	
22. $\frac{5a}{12b}+\frac{3x}{4}$	
23. $\frac{a}{px}+\frac{b}{qx}$	
24. $\frac{3a}{5}+\frac{b}{a}+\frac{2x}{3y}$	

25. Find the sum of $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, by substitution in Ex. 11.

26. Find the sum of $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, by substitution in Ex. 11.

74. VI. *To subtract fractions.* Reduce them, if necessary, to equivalent ones having a common denominator, and under the difference of the numerators place the common denominator.

Let it be required to find the difference of $\frac{a}{b}$ and $\frac{c}{d}$.

Reducing them to the same denominator, we have

$$\frac{ad}{bd} \text{ and } \frac{bc}{bd};$$

then

$$\frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}.$$

The reason of the rule may be explained in the same manner as that of addition.

EXERCISES.

Fractions to be subtracted.		Answers.
1.	$\frac{5x}{7a} - \frac{3y}{7}$	$\frac{5x-3ay}{7a}$.
2.	$\frac{a+b}{2} - \frac{a-b}{2}$	b.
3.	$\frac{3x}{7} - \frac{5x}{14}$	$\frac{1}{14}x$.
4.	$\frac{4x}{5} - \frac{3x}{4}$	$\frac{1}{20}x$.
5.	$\frac{p+q}{p-q} - \frac{p-q}{p+q}$	$\frac{4pq}{p^2-q^2}$.
6.	$\frac{n-1}{n} - \frac{n}{n-1}$	$\frac{1-2n}{n^2-n}$.

Fractions to be subtracted.		Answers.
7.	$\frac{1}{a-b} - \frac{1}{a+b}$	$\frac{2b}{a^2-b^2}$.
8.	$\frac{4a}{b} - \frac{4a-x}{b}$	$\frac{x}{b}$.
9.	$\frac{7n}{2} - \frac{2n+1}{3}$	$\frac{17n-2}{6}$.
10.	$\frac{1}{1-x} - \frac{2}{1-x^2}$	$\frac{x-1}{1-x^2} = -\frac{1}{1+x}$.
11.	$\frac{1}{x-2} - \frac{1}{x-5}$	$-\frac{3}{x^2-7x+10}$.
12.	$\frac{1}{x-6} - \frac{1}{x-5}$	$\frac{1}{x^2-11x+30}$.

75. VII. *To reduce a mixed quantity to a fractional form.* Reduce the integral part to a fraction, having the same denominator as the fractional part (Art. 69); and under the sum or difference of the numerators, put the common denominator.

The reason is obvious from Art. 73, 74.

EXAMPLES.

1. Reduce $a \pm \frac{b}{c}$ to a fractional form.

Multiplying the integral part, a, by the denominator, c, and adding or subtracting b according to the sign, we get $ac \pm b$; then writing c under this as a divisor, the result is $\frac{ac \pm b}{c}$.

$$a + \frac{b}{c}$$
$$\frac{ac+b}{c}$$
$$a - \frac{b}{c}$$
$$\frac{ac-b}{c}$$

2. Reduce $a - \frac{b-c}{d}$ to a fractional form.

Here, after multiplying, the compound numerator $b-c$ must be subtracted (Art. 33).

$$a-\frac{b-c}{d}$$
$$\frac{ad-(b-c)}{d}$$
$$\frac{ad-b+c}{d}$$

EXERCISES.

Quantities to be reduced.	Answers.
1. $x-\frac{y}{b}$	$\frac{bx-y}{b}$.
2. $a-7-\frac{-50+2a^2}{a+7}$	$\frac{1-a^2}{a+7}$.
3. $3b+\frac{4b-6}{5}$	$\frac{19b-6}{5}$.
4. $5x-2y^2+\frac{4xy-3y^3}{3y}$	$\frac{19xy-9y^3}{3y}$.
5. $7-\frac{4x+3}{5x}$	$\frac{31x-3}{5x}$.
6. $a+b+\frac{5a-2b^2}{2b}$	$\frac{2ab+5a}{2b}$.
7. $1-a-\frac{2x-a^2}{1+a}$	$\frac{1-2x}{1+a}$.
8. $1+\frac{b-x+7}{x}$	$\frac{b+7}{x}$.
9. $b+y-\frac{b^2-y^2-2}{b-y}$	$\frac{2}{b-y}$.
10. $2x-\frac{5}{12}x$	$\frac{19}{12}x$.
11. $x-y-\frac{x^2-y^2-5}{x+y}$	$\frac{5}{x+y}$.
12. $a-\frac{5}{4}\cdot\frac{a}{b}$	$\frac{a(4b-3)}{4b}$.
13. $1+x+\frac{x^2}{1-x}$	$\frac{1}{1-x}$.

Quantities to be reduced.	ANSWERS.
14. $1+x+x^2+\frac{x^3}{1-x}$	$\frac{1}{1-x}$.
15. $a+x-\frac{1}{3}(a-x)$	$\frac{2}{3}(a+2x)$.
16. $m-x-\frac{m^2-2mx+6+x^2}{m-x}$	$\frac{-6}{m-x}$.
17. $a-\frac{1}{a}$	$\frac{1}{a}(a^2-1)$.
18. $a-\frac{3}{4}(a-x)$	$\frac{1}{4}(a+3x)$.
19. $5ax-3by-\frac{10a^2x-6aby}{2a}$	0.

76. VIII. *To multiply fractions.* Muliply the numerators together for a numerator, and the denominators for a denominator.

A fraction may be multiplied by an integer, either by expressing the integer as a fraction (Art. 69), and applying the above rule, or by applying Art. 65.

Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

First, multiply $\frac{a}{b}$ by c, and the product is $\frac{ac}{b}$ (Art. 65): but this muliplier, c, is d times too large, therefore the product $\frac{ac}{b}$ is d times too large; it must therefore be divided by d, that is, the denominator must be multiplied by d (Art. 66), which gives $\frac{ac}{bd}$ for the product of the two fractions.

The process may often be shortened by suppressing, while performing the multiplication, those factors which are common to the numerator and denominator; as this

is, in fact, anticipating the reduction of the resulting fraction to its lowest terms.*

EXAMPLES.

1. Multiply $\frac{3a}{4b}$ by $\frac{2x}{3y}$.

$$\frac{3a}{4b} \times \frac{2x}{3y} = \frac{6ax}{12by} = \frac{ax}{2by}.$$

The same result will be at once obtained by suppressing the factors 3 and 2, which are common to both terms: thus,

$$\frac{3 \times 2 \times ax}{3 \times 4 \times by} = \frac{ax}{2by}.$$

2. Multiply together the fractions $\frac{nx}{3p}, \frac{2a}{y}, \frac{3b}{2a}$.

Here, multiplying the three numerators continually together, and also the three denominators, and at the same time suppressing the common factors $2a$ and 3, we obtain

$$\frac{bnx}{py}.$$

3. Multiply $\frac{3a}{12bx}$ by $3b$.

Here, the denominator is divisible by the integer, therefore (Art. 65) the product is

$$\frac{3a}{4x},$$

which is irreducible. If we had multiplied the numerator,

* The reason of the rule may be shown otherwise thus: let $\frac{a}{b} = x$, $\frac{c}{d} = y$; then, by the nature of division, $a = bx$, $c = dy$; therefore (Art. 29) $ac = bdxy$. Hence (Art. 30) $\frac{ac}{bd} = xy$; but $x = \frac{a}{b}$, $y = \frac{c}{d}$, therefore $\frac{ac}{bd} = \frac{a}{b} \times \frac{c}{d}$.

we should have had $\frac{9ab}{12bx}$;

which, being reduced, is $\frac{3a}{4x}$

If $3b$ be expressed as a fraction (Art. 69), we have

$$\frac{3a}{12bx} \times \frac{3b}{1} = \frac{9ab}{12bx} = \frac{3a}{4x}.$$

which is the same as multiplying the numerator by the integer.

4. Multiply $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}+\frac{x}{2}+\frac{x}{8}$ by the least common multiple of the denominators.

The least common multiple is plainly 24: multiplying the numerators by this number, and performing the actual division by the denominators (Art. 70), we obtain

$$8x+6x+4x+12x+3x=33x.$$

EXERCISES.

Fractions to be multiplied.	ANSWERS.
1. $\frac{a}{6} \times \frac{2a^2}{9}$	$\frac{a^3}{27}$.
2. $\frac{b}{c} \times \frac{x}{n} \times \frac{a}{y}$	$\frac{abx}{cny}$.
3. $\frac{2x}{5} \times \frac{4x^2}{5a} \times 3x$	$\frac{24x^4}{25a}$ or $\frac{24}{25} \cdot \frac{x^4}{a}$.
4. $\frac{a-x}{b-y} \times \frac{3x-1}{4a}$	$\frac{3x(a-x)-a+x}{4a(b-y)}$.
5. $\frac{2}{3}a \times \frac{3}{4}b$	$\frac{1}{2}ab$.
6. $\frac{x-1}{b-4} \times \frac{b-4}{2a}$	$\frac{x-1}{2a}$.
7. $\frac{a^2-b^2}{a} \times \frac{a}{a+b} \times \frac{1}{a-b}$	1.

Fractions to be multiplied.	Answers.
8. $\frac{3q^2-q}{5} \times \frac{10}{2q^2-4q}$	$\frac{3q-1}{q-2}$.
9. $\frac{a+b}{2} \times \frac{a-b}{2}$	$\frac{a^2-b^2}{4}$.
10. $\frac{a^2+b^2}{a^2-b^2} \times \frac{a-b}{a+b}$	$\frac{a^2+b^2}{(a+b)^2}$.
11. $\frac{a^n-x^n}{4a-2} \times \frac{a-x}{b^n+x^n} \times \frac{2a-1}{b^n-x^n}$	$\frac{a^{n+1}-ax^n-a^nx+x^{n+1}}{2(b^{2n}-x^{2n})}$.
12. $(a+\frac{1}{a}) \times (a-\frac{1}{a})$	$a^2-\frac{1}{a^2}$.
13. $\frac{a+x}{a-x} \times \frac{a-x}{a+x}$	1.
14. $(2a-\frac{1}{4}) \times (2a-\frac{1}{4})$	$4a^2-a+\frac{1}{16}$.
15. $\frac{x^2-9x+20}{x^2-6x} \times \frac{x^2-13x+42}{x^2-5x}$	$\frac{x^2-11x+28}{x^2}$.
16. $(\frac{3a}{4x}+b) \times (\frac{5a}{3x}-2b)$	$\frac{5a^2}{4x^2}+\frac{ab}{6x}-2b^2$.
17. $(\frac{1}{3}-3a) \times (\frac{2}{3}-6a)$	$\frac{2}{9}-4a+18a^2$.
18. $\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12}$	$\frac{x+2}{x+3}$.
19. $b \times \frac{1}{b}$	1.
20. $\frac{a}{a^2-x^2} \times (a-x)$	$\frac{a}{a+x}$.
21. $\frac{3}{11}a \times 11$	$3a$.
22. $\frac{2}{3}ax \times 3c$	$2acx$.

Fractions to be multiplied.	ANSWERS.
23. $\frac{1}{2}a \times 4$	$2a$.
24. $4ax \times \frac{3}{4}b$	$3abx$.
25. $\frac{1}{8}x \times 4$	$\frac{1}{2}x$.
26. $\frac{n-1}{n+1} \times (n+1)$	$n-1$.
27. $\frac{2ay}{15n^2x} \times 5n$	$\frac{2ay}{3nx}$.
28. $\frac{5bx}{4a} \times 4a$	$5bx$.
29. $(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{pqr}) \times pqrs$ ..	$qrs+prs+pqs-s$.

30. $(\frac{1}{2}a+\frac{1}{4}b+\frac{1}{5}c) \times$ least common multiple of the denominators. *Ans.* $10a+5b+4c$.

31. $(\frac{x}{3}+\frac{x}{12}-\frac{x}{5}+\frac{x}{20}-\frac{x}{4}+\frac{x}{15}+\frac{x}{30}) \times$ least common multiple of the denominators. *Ans.* $7x$.

32. $(\frac{x}{5}+\frac{x}{6}+x) \times$ least common multiple of the denominators. *Ans.* $41x$.

77. IX. *To divide by a fraction.* Invert the divisor, and proceed as in multiplication.*

* If both divisor and dividend be fractions, and if each term of the divisor measure the corresponding term of the dividend, the quotient may be obtained by dividing the numerator and denominator of the dividend by the numerator and denominator of the divisor, respectively. This rule is, however, so seldom applicable, that it is of no practical utility. It may be proved in the same manner as the rule in Art. 76.

A fraction may be divided by an integer, either by expressing the integer as a fraction (Art. 69), and applying the above rule, or by applying Art. 66.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$

First divide $\frac{a}{b}$ by c, and the quotient is $\frac{a}{bc}$ (Art. 66); but this divisor is d times too large, therefore the quotient $\frac{a}{bc}$ is d times too small: it must, therefore, be multiplied by d, which gives $\frac{ad}{bc}$. Now $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$, or $\frac{a}{b}$ multiplied by the divisor inverted.*

78. *Cor.* Hence, unity divided by a fraction is the same as that fraction inverted; for $1 \div \frac{b}{a} = 1 \times \frac{b}{a} = \frac{b}{a}$.

79. Def. Unity divided by any quantity is called the *reciprocal* of that quantity: thus $\frac{1}{x}$ is the reciprocal of x, and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$. Since an integral quantity may be considered as a fraction whose denominator is 1, the reciprocal of any quantity, integral as well as fractional, may be considered as that quantity inverted.

80. *Cor.* Hence, to divide by any quantity is the same as to multiply by its reciprocal, and *vice versa*.

* Otherwise thus: let $\frac{a}{b} = x$, $\frac{c}{d} = y$; then, by the nature of division, $a = bx$, $c = dy$; therefore (Art. 29) $ad = bdx$, $bc = bdy$. Also, $\frac{ad}{bc} = \frac{bdx}{bdy} = x \div y = \frac{a}{b} \div \frac{c}{d}$.

EXAMPLES.

1. Divide $\frac{3a}{5b}$ by $\frac{2x}{3c}$.

Here inverting the divisor, and multiplying, we have

$$\frac{3a}{5b} \times \frac{3c}{2x} = \frac{9ac}{10bx}.$$

2. Divide $\frac{9ab}{10cd}$ by $3b$.

Here, dividing the numerator by $3b$ (Art. 66), we have,

$$\frac{3a}{10cd}$$

If $3b$ be expressed as a fraction, we shall have

$$\frac{9ab}{10cd} \div \frac{3b}{1} = \frac{9ab}{10cd} \times \frac{1}{3b} = \frac{9ab}{30bcd} = \frac{3a}{10cd},$$

which is the same as applying the second part of the Prop. in Art. 66.

3. Divide $a - \frac{b}{c}$ by $x + \frac{d}{y}$.

Reducing these to a fractional form (Art. 75), and dividing, we have

$$\frac{ac-b}{c} \div \frac{xy+d}{y} = \frac{(ac-b)y}{(xy+d)c}.$$

EXERCISES.

Fractions to be divided.	ANSWERS.
1. $\frac{4x}{5} \div \frac{9x}{10}$	$\frac{8}{9}$.
2. $\frac{1}{m+q} \div \frac{1}{m-q}$	$\frac{m-q}{m+q}$.
3. $a \div \frac{b}{c}$	$\frac{1}{b} \cdot ac$
4. $1 \div \frac{1}{a}$	a.

Fractions to be divided.	ANSWERS.
5. $\frac{n^2-9}{x-1} \div \frac{n+3}{x^2-2x+1}$	$(n-3)(x-1)$.
6. $3x \div \frac{4a^2}{5x}$	$\frac{15x^2}{4a^2}$.
7. $\frac{1}{x} \div \frac{1}{x^2}$	x.
8. $\frac{a}{1-a} \div \frac{a}{4}$	$\frac{4}{1-a}$.
9. $a \div \frac{1}{a}$	a^2.
10. $\frac{b}{3} \div \frac{1}{3}b$	1.
11. $\frac{1}{10} \div \frac{1}{2}x$	$\frac{1}{5} \cdot \frac{1}{x}$.
12. $4 \div \frac{4}{a}$	a.
13. $\frac{1}{5} \div \frac{1}{x}.10$	$\frac{x}{50}$.
14. $\frac{a+b}{2} \div \frac{a-b}{2}$	$\frac{a+b}{a-b}$.
15. $\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b}$	$\frac{a^2+b^2}{a^2-2ab+b^2}$.
16. $\frac{x^2+3x+2}{x^2+2x+1} \div \frac{x^2+5x+4}{x^2+7x+12}$	$\frac{x^2+5x+6}{x^2+2x+1}$.
17. $\frac{\frac{a}{4}}{\frac{b}{3}}$	$\frac{3}{4} \cdot \frac{a}{b}$.
18. $\frac{x}{\frac{1}{y}}$	xy.

Fractions to be divided.	Answers.
19. $\dfrac{\frac{m}{n}}{\frac{p}{q}}$	$\dfrac{mq}{np}$.
20. $\dfrac{1+x}{1+\frac{1}{x}}$	x.
21. $\dfrac{1}{\frac{1}{x}}$	x.
22. $\dfrac{1+x}{\frac{1}{x}(1+x)}$	x.
23. $\dfrac{1+\frac{1}{a}}{1+\frac{1}{a^2}}$	$\dfrac{a^2+a}{a^2+1}$.
24. $\dfrac{1}{1+\frac{1}{x}}$	$\dfrac{x}{1+x}$.
25. $\dfrac{a}{b+\frac{c}{d}}$	$\dfrac{ad}{bd+c}$.
26. $\dfrac{\frac{1}{1+x}}{1-\frac{1}{1+x}}$	$\dfrac{1}{x}$.

Fractions to be divided.	Answers.
27. $\dfrac{\frac{1}{1+x}+\frac{x}{1-x}}{\frac{1}{1-x}-\frac{x}{1+x}}$	1.
28. $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}$	1.
29. $\dfrac{\frac{x-4}{2}}{2\frac{1}{2}}$	$\frac{1}{5}(x-4)$.
30. $\dfrac{b+\frac{1}{2}}{\frac{2}{3}b+\frac{1}{4}}$	$\frac{12b+6}{8b+3}$.
31. $\dfrac{a}{b+\frac{c}{d+\frac{e}{f}}}$*	$\frac{a(df+e)}{(bd+c)f+be}$.
32. $\dfrac{1}{1+\frac{1}{1+\frac{1}{x}}}$	$\frac{1+x}{1+2x}$.
33. $\frac{a}{b} \div c$	$\frac{a}{bc}$.

* The most easy way of working this exercise, is to substitute $d+\frac{e}{f}$ in place of d, in exercise 25.

Fractions to be divided.	Answers.
34. $\frac{4a}{5b} \div 2a$	$\frac{2}{5b}$.
35. $\frac{1}{b} \div b$	$\frac{1}{b^2}$.
36. $\frac{n^2-9}{a+3} \div (n-3)$	$\frac{n+3}{a+3}$.
37. $\frac{1}{2}b \div 4b$	$\frac{1}{8}$.
38. $\frac{2}{3}ax \div 4bx$	$\frac{1}{6}\frac{a}{b}$.
39. $\frac{\frac{1}{b-x}}{b+x}$	$\frac{1}{b^2-x^2}$.
40. $(a^4-\frac{1}{a^4}) \div (a-\frac{1}{a})$	$a^3+\frac{1}{a^3}+a+\frac{1}{a}$.
41. $(x^6+\frac{1}{x^6}+x^4+\frac{1}{x^4}+x^2+\frac{1}{x^2}+2) \div (x^3+\frac{1}{x^3}+x+\frac{1}{x})$	$x^3+\frac{1}{x^3}$.
42. $\frac{1}{2} \div a$	
43. $\frac{3}{4}a \div 3b$	
44. $1\frac{1}{2}n^2x \div 3x$	
45. $2\frac{1}{3}\ a \div 3a$	

46. Divide a by its reciprocal.

47. Divide $\sqrt{x}$ by its reciprocal.

48. Divide $a-x$ by its reciprocal.

49. Divide $\frac{a-b}{c}$ by its reciprocal.

50. Divide $\frac{1}{a}$ by its reciprocal.

RESOLUTION OF FRACTIONS INTO INFINITE SERIES.

81. It has been already remarked, that a fraction indicates a division which cannot be actually performed: but, when an algebraic fraction has a compound denominator, whether the numerator be simple or compound, an attempt to effect the division indicated will always give rise, not to a complete quotient, or one that can be expressed in a finite number of terms, but to a series of terms which may be continued without limit. Thus the fraction

$$\frac{x}{a+b},$$

by actual division of the numerator by the denominator, will give an infinite series for quotient:

$$\begin{array}{l|l}
a+b) \; x & \frac{x}{a} - \frac{bx}{a^2} + \frac{b^2x}{a^3} - \frac{b^3x}{a^4} \text{ \&c., or} \\
\quad x + \frac{bx}{a} & \\
\hline
\quad -\frac{bx}{a} & \frac{x}{a}\left(1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} \text{ \&c.,}\right) \\
\quad -\frac{bx}{a} - \frac{b^2x}{a^2} & \\
\hline
\quad \frac{b^2x}{a^2} & \\
\quad \frac{b^2x}{a^2} + \frac{b^3x}{a^3} & \\
\hline
\quad -\frac{b^3x}{a^3} & \\
\quad -\frac{b^3x}{a^3} - \frac{b^4x}{a^4} & \\
\hline
\quad \frac{b^4x}{a^4} \text{ \&c.} &
\end{array}$$

The remainder will never disappear, and consequently a complete quotient can never be found. The law which the quotient follows is, however, obvious; each successive term being found by multiplying the one before it by $\frac{b}{a}$, and the signs being alternately *plus* and *minus*.

The process may be verified in the usual manner, by multiplying the divisor by any part of the quotient, and adding in the remainder at that part. Thus, in the above example, by multiplying the first four terms by $a+b$, and adding in the remainder $\frac{b^4x}{a^4}$, the dividend x will be reproduced. And the same would be found by taking any other number of terms, and adding the corresponding remainder.

2. Divide $x+a$ by $x+b$.

$$x+b)x+a(1+\frac{a-b}{x}-\frac{(a-b)b}{x^2}+\frac{(a-b)b^2}{x^3}\ \&c.$$

$$\underline{x+b}$$

$$a-b$$

$$a-b+\frac{(a-b)b}{x}$$

$$\underline{\qquad\qquad}$$

$$\underline{\frac{(a-b)b}{x}}$$

$$-\frac{(a-b)b}{x}-\frac{(a-b)b^2}{x^2}$$

$$\underline{\qquad\qquad\qquad}$$

$$\underline{\frac{(a-b)b^2}{x^2}}$$

$$\frac{(a-b)b^2}{x^2}+\frac{(a-b)b^3}{x^3}$$

$$\underline{\qquad\qquad\qquad}$$

$$-\frac{(a-b)b^3}{x^3}\ \&c.$$

3. Reduce the fraction $\frac{1}{1+x}$ into an infinite series by division.

Here proceeding as in the other examples, we find for quotient, $1-x+x^2-x^3$, &c.

4. If the terms of the divisor, in the last example, be reversed, the resulting series will be

$$\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}\ \&c.$$

82. In example 3, last Art. let $x=\frac{1}{3}$; then

$$\frac{1}{1+x}=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\frac{3}{4}=1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}\ \&c.$$

If we consider only two terms of this series, the sum will be $\frac{2}{3}$, which is less than the true sum by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which differs from the true sum by $\frac{1}{36}$; four terms make $\frac{20}{27}$ which differs only by $\frac{1}{108}$, &c. Hence we might infer that if the series were continued to infinity, the difference would disappear, and the sum would be $\frac{3}{4}$.

In the series of the first example, let $a=6$, $b=4$, $x=3$; then

$$\frac{x}{a+b}=\frac{3}{10}=\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\frac{4}{27},\ \&c.$$

And, if the first two terms be taken, the sum is too small by $\frac{2}{15}$; if three, it is too large by $\frac{4}{45}$; and so on. At last, therefore, the sum will be $\frac{3}{10}$.

83. Similar series sometimes arise in Division: thus, if

the terms of the dividend of the first example in Art. 44 be arranged as under, and the arrangement be adhered to, the quotient will be an infinite series. *See Note*, p. 46.

$$a+x)2ax+x^2+a^2(2x-\frac{x^2}{a}+\frac{x^3}{a^2}-\frac{x^4}{a^3}, \text{ \&c.}$$

$$\begin{array}{l} 2ax+2x^2 \\ \hline -x^2+a^2 \\ -x^2-\frac{x^3}{a} \\ \hline \frac{x^3}{a}+a^2 \\ \frac{x^3}{a}+\frac{x^4}{a^2} \\ \hline -\frac{x^4}{a^2}+a^2, \text{ \&c.} \end{array}$$

The remainders in the third case of Division, may also be expanded into series of the same kind. Thus, in example 2, $\frac{2a^4}{x+}$ may be so expanded.

$$x+a)2a^4 \quad \left(2(\frac{a^4}{x}-\frac{a^5}{x^2}+\frac{a^6}{x^3}-\frac{a^7}{x^4}, \text{ \&c.}\right.$$

$$\begin{array}{l} 2a^4+2\frac{a^5}{x} \\ \hline -2\frac{a^5}{x} \\ -2\frac{a^5}{x}-2\frac{a^6}{x^2} \\ \hline 2\frac{a^6}{x^2} \\ 2\frac{a^6}{x^2}+2\frac{a^7}{x^3} \\ \hline -2\frac{a^7}{x^3} \\ -2\frac{a^7}{x^3}-2\frac{a^8}{x^4} \\ \hline 2\frac{a^8}{x^4} \text{ \&c.} \end{array}$$

or

$$2\frac{a^4}{x}(1-\frac{a}{x}+\frac{a^2}{x^2}-\frac{a^3}{x^3}+\frac{a^4}{x^4}\text{ \&c.})$$

The quotient is therefore interminable, whenever there is such a remainder. *See page* 46.

EXERCISES.*

1. Divide 1 by $1-x$. *Ans.* $1+x+x^2+x^3$ &c.

2. ..... 4 by $1-a$.

Ans. $4(1+a+a^2+a^3$ &c.)

3 ..... a by $a+x$.

Ans. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}$ &c.

4. ..... a by $1+x$.

Ans. $a(1-x+x^2-x^3$ &c.)

5. ..... a by $x+1$.

Ans. $a(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}$ &c.)

6. ..... $a-x$ by $b-x$.

Ans. $\frac{a}{b}+\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}+\frac{(a-b)x^3}{b^4}$ &c.

7. ..... $x+2$ by $x+3$.

Ans. $1-\frac{1}{x}+\frac{3}{x^2}-\frac{3^2}{x^3}+\frac{3^3}{x^4}-\frac{3^4}{x^5}$ &c.

8. ..... 1 by x^2-2x+1.

Ans. $\frac{1}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}+\frac{4}{x^5}+\frac{5}{x^6}+\frac{6}{x^7}$ &c.

9. ..... $1-3x-2x^2$ by $1-4x$.

Ans. $1+x+2x^2+2.4.x^3+2.4^2.x^4$ &c.

10. ..... 1 by $1-2x+x^2$.

Ans. $1+2x+3x^2+4x^3+5x^4+6x^5$ &c.

11. What will $\frac{1}{x}$ become, when x is changed into $x+h$.

Ans. $\frac{1}{x}-h.\frac{1}{x^2}+h^2.\frac{1}{x^3}-h^3.\frac{1}{x^4}$ &c.

* These should be illustrated with particular numbers, as in Art. 82.

CHAP. IV.—POWERS.

ART. 84. Since (Art. 41) $\frac{a^n}{a^n}=a^{n-n}=a^0$, and also $\frac{a^n}{a^n}=1$, it follows that $a^0=1$. *Hence, that power of any quantity, whose index is zero, is equal to unity.*

Again, since $a^0=1$, if we divide each of these equals by a^n, we obtain (Art. 30)

$$a^{0-n}, \text{ or } a^{-n}=\frac{1}{a^n}.$$

Hence, any power with a negative index, is equal to the reciprocal of the same power with a positive index.

85. It follows from Art. 84, 29, 80, that

$$\frac{b}{a^n x}=\frac{b}{x}.\frac{1}{a^n}=\frac{b}{x}.a^{-n}=\frac{ba^{-n}}{x},$$

and

$$\frac{a^n b}{x}=\frac{b}{x}a^n=\frac{b}{x}\div\frac{1}{a^n}=\frac{b}{x}\div a^{-n}=\frac{b}{a^{-n}x};$$

that is to say, that a quantity may be taken from the denominator into the numerator of a fraction, and from the numerator into the denominator, by changing the sign of its index.

86. The square root may be expressed as a power, by using an index, which, when added to itself, will give 1: this index must be $\frac{1}{2}$; and then $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}=a^1=a$: hence $a^{\frac{1}{2}}=\sqrt{a}$. In the same manner, since $\sqrt[3]{a}\times\sqrt[3]{a}\times\sqrt[3]{a}=a=a^1$, and $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$, we may represent the cube root by the index $\frac{1}{3}$; then $a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=$

$a^1 = a$. And, in general, $\sqrt[n]{a} = a^{\frac{1}{n}}$, for $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}}$ &c. to n terms $= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\text{ &c. to } n \text{ terms}} = a^{\frac{n}{n}} = a^1 = a$. Again, since a quantity is squared, when its index is doubled, and since $\sqrt[3]{a} = a^{\frac{1}{3}}$, it follows that $\sqrt[3]{a^2} = a^{\frac{2}{3}}$; so that $a^{\frac{1}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{1}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{5}{3}} = a^2$. In like manner, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$; for $a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3$. And, in general, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, for $a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ &c. to n terms $= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}\text{ &c. to } n \text{ terms}} = a^{\frac{nm}{n}} = a^m$; the nth root of a^m being such, that it produces a^m when repeated n times as a factor.

87. PROB. I.—*To raise a quantity to any power.* Multiply the quantity continually by itself, until it has been used as a factor as many times as there are units in the index of the required power.

The reason of the rule is manifest from the nature of powers (Art. 14), involution being merely a succession of multiplications.

EXAMPLES.

1. Raise $-a^3b^2x^4$ to the third power.

By continual multiplication, we have

$$-a^3b^2x^4 \times -a^3b^2x^4 \times -a^3b^2x^4 = -a^9b^6x^{12}.$$

2. Involve $4a^{\frac{1}{3}}b^{\frac{2}{3}}$ to the second power.

Here $4a^{\frac{1}{3}}b^{\frac{2}{3}} \times 4a^{\frac{1}{3}}b^{\frac{2}{3}} = 16ab^{\frac{4}{3}} = 16a\sqrt[3]{b^4}$.

3. Raise $\frac{3a^3}{4x^{\frac{1}{2}}}$ to the fourth power.

Multiplying continually, we have

$$\frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} = \frac{81a^{12}}{256x^2}.$$

4. Involve $a+b$ to the third power.

Multiplying $a+b$ by itself, and the product $a^2+2ab+b^2$ again by $a+b$, we obtain

$$a^3+3a^2b+3ab^2+b^3$$

for the required power. This may be put under the form $a^3+b^3+3ab(a+b)$.

88. *Cor.* 1. Hence, a quantity may be raised to any power, by multiplying its index by the index of the power; that is, $(a^m)^n = a^{mn}$.

For the continued product of a^m into itself n times, is found by adding the index m (Art. 35) as often as there are units in n; that is, multiplying m by n.

89. *Cor.* 2. Hence, also, any power of a product is equal to the product of the same powers of its factors, that is, $(ab)^n = a^n b^n$; for $(ab)^n = ab \times ab \times ab$ &c. to n terms $= a \times a \times a$ &c. to n terms $\times b \times b \times b$ &c. to n terms (Art. 11) $= a^n b^n$.

90. *Cor.* 3. Hence also it follows that, when the quantity to be involved is negative, all the *even* powers will be positive, and all the *odd* powers negative.

91. According to the preceding rule, the successive powers of the binomial $a+b$, found by continual multiplication, will be as follows:—

$(a+b)^2 = a^2+2ab+b^2$.

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

$(a+b)^7 = a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

By comparing these developments it appears that,

1. The index of a decreases, and that of b increases, by a unit in each successive term; so that the sum of the indices in each term is always equal to the index of the

given power. Hence, in general, the terms of $(a+b)^n$, without the coefficients, will be

$a^n, a^{n-1}b, a^{n-2}b^2, a^{n-3}b^3, a^{n-4}b^4, a^{n-5}b^5$, &c. .. ab^{n-1}, b^n.*

2. The number of terms is one greater than the index; for there must be as many powers of a as there are units in the index, and one term besides, which does not contain a; so that when the index is even, the number of terms is odd, and *vice versa*.

3. The coefficients increase as far as the middle term, and then diminish, the terms equidistant from it having the same coefficient: thus those of the first and last are unity, of the second and last but one, the same as the given index, &c. And in general, if we know the coefficient in any term, that of the next following will be found by multiplying this coefficient by the index of a in that term, and dividing the product by the number of terms. Thus, in the case of $(a+b)^6$, the coefficient of the second term is 6, the index of a in that term is 5, the number of terms is 2; and $\frac{6\times5}{2} = 15$, the coefficient of the third term. In like manner, $\frac{15\times4}{3} = 20$, the coefficient of the fourth term; and on examining the coefficients of the other powers, we shall find that they may all be formed in the same manner.

According to this law, the coefficients of the terms of $(a+b)^n$ will be as follows:—

The coefficient of the first term ..... 1.

The coefficient of the second term ..... $\frac{1\times n}{1} = n$.

* The first and last terms may (Art. 84), be written $a^n b^0$, and $a^0 b^n$.

The coefficient of the third term ..... $\frac{n(n-1)}{2}$.

The coefficient of the fourth term

$$\frac{n(n-1)}{2} \times \frac{n-2}{3} = \frac{n(n-1)(n-2)}{2 \cdot 3}.$$

The coefficient of the fifth term

$$\frac{n(n-1)(n-2)}{2 \cdot 3} \times \frac{n-3}{4} = \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}.$$

&c. &c.

Combining these with the terms of $(a+b)^n$, already given, we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)}{2}\frac{(n-2)}{3}$$

$$a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}a^{n-4}b^4 + \frac{n(n-1)}{2}\frac{(n-2)}{3}$$

$$\frac{(n-3)(n-4)}{4 \cdot 5}a^{n-5}b^5 + \text{&c.} \ldots\ldots\ldots\ldots + nab^{n-1} + b^n.$$

This is the celebrated BINOMIAL THEOREM, by means of which a binomial quantity may be involved to any power without the trouble of continual multiplication.*

92. If the second term, b, of the binomial be negative, its even powers will be positive, and its odd powers negative (Art. 90); but the even terms of the expanded power contain the odd powers of b, and therefore the powers of a being all positive, the odd terms of the powers of $a-b$ will be positive, and the even terms negative.

* A general demonstration of the binomial theorem will be given in a treatise on the principles of the higher Algebra, which will be published as a sequel to this work. It is convenient to introduce the theorem here for practical purposes; and the above inductive proof is the less objectionable, as no subsequent demonstration is founded upon it, and as it seems to have been by a similar process that Newton himself inferred the generality of this important formula.

In other respects, the powers of $(a-b)^n$ will be the same as those of $(a+b)^n$.

Further, it is plain from the manner in which the developments are generated, that the series will terminate at the $(n+1)th$ term, when n is a whole positive number, since, on passing that term, $n-n$ or zero would always occur as a factor, and therefore every higher term would vanish. But assuming that the series holds when n is negative or fractional, n will not, in this case, be destroyed by any of the numbers successively taken from it. Thus no factor of the coefficients can become zero, and therefore the series will never terminate.

EXAMPLES.

1. Required the 6th power of $2x^2-3y$.

In this case we must put $2x^2,-3y$, and 6, for a, b, and n, in the general formula. Hence,

The first term is ..... $(2x^2)^6=64x^{12}$.
The second ..... $6\times(2x^2)^5\times(-3y)=-576x^{10}y$.
The third ..... $6\times\frac{5}{2}\times(2x^2)^4\times(-3y)^2=2160x^8y^2$.
The fourth .... $15\times\frac{4}{3}\times(2x^2)^3\times(-3y)^3=-4320x^6y^3$.
The fifth ..... $20\times\frac{3}{4}\times(2x^2)^2\times(-3y)^4=4860x^4y^4$.
The sixth ..... $15\times\frac{2}{5}\times 2x^2\times(-3y)^5=-2916x^2y^5$.
The seventh ..... $6\times\frac{1}{6}\times(2x^2)^0\times(-3y)^6=729y^6$.

Combining these, we have

$$(2x^2-3y)^6=64x^{12}-576x^{10}y+2160x^8y^2-4320x^6y^3+4860x^4y^4-2916x^2y^5+729y^6.$$

2. It is required to expand $\frac{x}{(a+b)^2}$ by the binomial theorem.

Since (Art. 85) $\frac{x}{(a+b)^2}=x\,(a+b)^{-2}$, we shall first expand $(a+b)^{-2}$, and then multiply all the terms by x.

Here $n = -2$, therefore

The first term is ..... a^{-2}.

The second ..... $-2a^{-3}b$.

The third ..... $\frac{-2\times-3}{2}a^{-4}b^2=3a^{-4}b^2$.

The fourth ..... $3\times\frac{-4}{3}a^{-5}b^3=-4a^{-5}b^3$.

The fifth ..... $-4\times\frac{-5}{4}a^{-6}b^4=5\,a^{-6}b^4$.

&c. &c.

That the series will never terminate may also be shown by actual division; for (Art. 81) $(a+b)^{-2}$, or $\frac{1}{(a+b)^2}$. may be expanded into an infinite series.

We have therefore

$(a+b)^{-2}=a^{-2}-2a^{-3}\,b+3a^{-4}b^2-4\,a^{-5}b^3+5\,a^{-6}\,b^4$ &c.

or (Art. 85)

$$\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}\text{ &c.}$$

Hence,

$$\frac{x}{(a+b)^2}=\frac{x}{a^2}\left(1-\frac{2b}{a}+\frac{3b^2}{a^2}-\frac{4b^3}{a^3}+\frac{5b^4}{a^4}\text{ &c.}\right)$$

Precisely the same series would be found by dividing x by $a^2+2ab+b^2$ (Art. 81).

3. Raise $a+b+c$ to the third power.

Regarding $a+b$ as a single term, we may express $(a+b+c)^3$ under the form $\{(a+b)+c\}^3$; then

The first term is ..... $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

The second ..... $3(a+b)^2c=3a^2c+6abc+3b^2c$.

The third ..... $\frac{3+2}{2}(a+b)c^2=3ac^2+3bc^2$.

The fourth..... ..... $\frac{3\times1}{3}c^3=c^3$.

Therefore,

$[(a+b)+c]^3 = (a+b+c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 = (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$;

which latter form is the same as that of the cube of a binomial.

EXERCISES.

	ANSWERS.
1. Multiply $3a^{-4}$ by $5a^6$	$15a^2$.
2. Divide a^3 by a^{-4}	a^7.
3. Express $6a^{-2}+b^2x^{-3}$ with positive indices	$\frac{6}{a^2}+\frac{b^2}{x^3}$.
4. Express $\frac{bx^2}{a^3y^{-2}}$ without a denominator ..	$a^{-3}bx^2y^2$.
5. Divide $12b^4x^{-2}$ by $4b^{-3}x^4$	$\frac{3b^7}{x^6}$.
6. Express $ax^{-(n-1)}$ as a fraction	$\frac{a}{x^{n-1}}$.
7. Express $\frac{b}{a^{4-n}}$ without a denominator	ba^{n-4}.
8. Divide $x^{\frac{1}{3}}$ by x	$\frac{1}{\sqrt[3]{x}}$.
9. Express $\frac{b}{x\sqrt[4]{x}}$ without a denominator	$bx^{-\frac{5}{4}}$.
10. Divide $x^{-\frac{5}{4}}$ by x	$\frac{1}{\sqrt[4]{x^9}}$.
11. Multiply $\frac{1}{2}a^{-n}$ by a^n	$\frac{1}{2}$.
12. Raise $-2a^3b^{\frac{1}{2}}$ to the fourth power	$16a^{12}b^2$.
13. Raise $\sqrt[5]{a^2}$ to the sixth power	$\sqrt[5]{a^{12}}$.
14. Raise $\sqrt[3]{(a+x)}$ to the sixth power	$a^2+2ax+x^2$.
15. Raise $4a^5b$ to the nth power	$4^na^{5n}b^n$.

16. Express $\frac{1}{2}(1-x)(1+x^2)^{-\frac{1}{2}} \times 2x$ with a positive index.

Ans. $\frac{x(1-x)}{\sqrt{(1+x^2)}}$.

17. Raise a—6 to the 6th power.

Ans. $a^6-36a^5+540a^4-4320a^3+19440a^2-46656a+46656$.

18. Raise $5+4x$ to the 4th power.

Ans. $625+2000x+2400x^2+1280x^3+256x^4$.

19. Raise $1+x$ to the nth power.

Ans. $1+nx+n\frac{n-1}{2}x^2+n\frac{(n-1)(n-2)}{2\ .\ 3}x^3\ ..\ x^n$.

20. Expand $\frac{x}{a+b}$ by the binomial theorem.

Ans. $\frac{x}{a}(1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}$ &c.)

21. Expand $\frac{1}{(a+2b)^3}$ by the binomial theorem.

Ans. $\frac{1}{a^3}(1-\frac{6b}{a}+\frac{24b^2}{a^2}-\frac{80b^3}{a^3}$ &c.)

22. Find the square of the sum of x and its reciprocal.

Ans. $x^2+\frac{1}{x^2}+2$.

23. Express $x^{-\frac{2}{3}}$ with a positive index .. *Ans.* $\sqrt[3]{(\frac{1}{x})^2}$.

24. Raise x^2+3y^2 to the 5th power.

Ans. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

25. Raise $a-x$ to the 9th power.

Ans. $a^9-9a^8x+36a^7x^2-84a^6x^3+126a^5x^4-126a^4x^5+84a^3x^6-36a^2x^7+9ax^8-x^9$.

26. Expand $\frac{2}{(a+x)^2}$. *Ans.* $\frac{2}{a^2}(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+$&c)

27. Raise $1+x$ to the power whose index is $n+1$.

Ans. $1+(n+1)x+(n+1)\frac{n}{2}x^2+(n+1)\frac{n(n-1)}{2\ .\ 3}x^3+\frac{n(n-1)\ (n-2)}{2\ .\ 3\ .\ 4}x^4\ .....\ x^{n+1}$.

28. Required the 5th term of $(a^2-x^2)^{12}$.

Ans. $495a^{16}x^8$.

29. Required the middle term of $(a-x)^{16}$.

Ans. $12870a^8x^8$.

30. Required the coefficient of x^5 in the expansion of $(x+a)^8$. *Ans.* 56.

31. Expand $(1-x)^{-n}$ *Ans.* $1+nx+\frac{n(n+1)}{1\cdot 2}x^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3+$&c.

32. Required the 10th power of $a-1$.

Ans. $a^{10}-10a^9+45a^8-120a^7+210a^6-252a^5+210a^4-120a^3+45a^2-10a+1$.

33. Expand $\frac{1}{(1-x)^3}$. *Ans.* $1+3x+6x^2+10x^3+$ &c.

34. Expand $\frac{1}{(1+x^2)^3}$. *Ans.* $1-3x^2+6x^4-10x^6+$ &c.

35. Divide $\sqrt{x}$ by x.

36. Raise x^3 to the 4th power.

37. .... $\sqrt{x}$ to the 4th power.

38. .... $\frac{1}{a^5}$ to the 5th power.

39. .... $5\sqrt{a}$ to the 3d power.

40. .... $\frac{2}{3}a^n$ to the 4th power.

41. .... $\sqrt[4]{a^3}$ to the 4th power.

42. .... $4a^{-2}b^3$ to the 3d power.

43. .... $-\frac{1}{2}a^2$ to the 2d power.

44. .... $-a^2x^3$ to the 5th power.

45. .... $\frac{1}{2}\sqrt[3]{a^2}$ to the 3d power.

46. .... x^a to the power b.

47. .... $a^mb^nc^p$ to the power r.

48. .... $\sqrt[4]{x}$ to the 2d power.

49. Express $\frac{3}{2}a(1-x)^{-\frac{5}{4}}$ with a positive index.

50. ..... $a^2x^{-2}-ab^2x^{-3}+bx^{-1}$ with positive indices.

51. ..... $\frac{a^4}{x^3}+\frac{2a^3}{x^2}-\frac{ab}{x}$ without denominators.

52. Find the cube of $\sqrt[n]{a^m}$.

53. ..... 6th power of $3x+2y$.

54. Expand $\frac{1}{1+x}$, and $\frac{1}{(1+x)^2}$.

55. Square $x^{-1}-y^{-1}$, both in this form and with positive indices.

56. Raise $\sqrt[4]{(a+b)}$ to the 5th power.

57. ..... $1+\frac{1}{x}$ to the nth power.

58. Find the 2d, 3d, 4th, and 5th powers of $x+\frac{1}{x}$.

59. Find the 3d, 4th, and 5th powers of $1+x$, by substitution in Exercise 19.

60. Raise x^2-2x+1 to the 3d power.

EVOLUTION.

93. PROB. I. *To extract any root of a simple quantity.* Find what is the quantity which, when raised to the corresponding power, will produce the given quantity.

The reason of the rule is manifest from the nature of roots (Art. 15).

EXAMPLES.

1. Required the third root of $64a^6b^{12}$.

Here it is required to find of what quantity $64a^6b^{12}$ is the third power: the quantity is plainly $4a^2b^4$, because $(4a^2b^4)^3 = 64\,a^6\,b^{12}$ (Art. 87); therefore $\sqrt[3]{(64\,a^6\,b^{12})} = 4a^2b^4$.

2. Find the square root of $a^{-6}b^{\frac{1}{2}}x^3$.

Since $(a^{-3}\,b^{\frac{1}{4}}\sqrt{x^3})^2 = a^{-6}b^{\frac{1}{2}}x^3$, the required root is $a^{-3}b^{\frac{1}{4}}\sqrt{x^3} = \frac{b^{\frac{1}{4}}x^{\frac{3}{2}}}{a^3}$.

3. Find the cube root of $-27a^6b^{-9}x^3$.

Here $(-3\,a^2\,b^{-3}x^{\frac{1}{2}}) \times (-3\,a^2\,b^{-3}x^{\frac{1}{2}}) \times (-3\,a^2\,b^{-3}x^{\frac{1}{2}}) = -27a^6b^{-9}x^3$; and the required root is therefore $-3a^2\,b^{-3}x^{\frac{1}{2}}$.

4. Find the fourth root of the cube root of x.

Here, since $\sqrt[3]{x} = x^{\frac{1}{3}}$, the index of the required root must be such that, when it is added four times, it will produce $\frac{1}{3}$: it must be $\frac{1}{12}$, therefore $\sqrt[4]{(\sqrt[3]{x})} = x^{\frac{1}{12}} = \sqrt[12]{x}$.

94. *Cor.* 1. Hence, any root of a given power is extracted by dividing the index of the power by the index of the root, or $\sqrt[n]{a^m} = a^{\frac{m}{n}}$; for $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ &c. to n terms $= a^{\frac{mn}{n}} = a^m$.*

95. *Cor.* 2. Hence also, any root of a product is equal to the product of the same root of each of its factors; that is, $\sqrt[n]{(ab)} = \sqrt[n]{a} \times \sqrt[n]{b}$, because each of these, by involution, produces the quantity ab.

96. *Cor.* 3. Hence, and from Art. 90, it follows, that the even roots of all positive quantities may be either positive or negative; and that the odd roots of positive quantities are positive, and those of negative quantities negative; that is to say, $\sqrt[2n]{+a} = \pm\sqrt[2n]{a}$; $\sqrt[2n+1]{+a} = +\sqrt[2n+1]{a}$; and $\sqrt[2n+1]{-a} = -\sqrt[2n+1]{a}$. But

* From this Corollary we might have derived the method of representing roots as powers with fractional indices (Art. 86); for $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, the denominator expressing the root, and the numerator the power whose root is taken.

an even root of a negative quantity is not assignable by any process of evolution: for we know that $(+a)\times(+a) = +a^2$, and that $(-a)\times(-a) = +a^2$, and, therefore, that $\sqrt{a^2} = \pm a$; but we know of no quantity which, multiplied by itself, will produce $-a^2$: and the same must hold whenever the index of the root is even: therefore, $\sqrt[n]{-a}$ is not assignable.

EXERCISES.

Quantities to be evolved.	Answers.
1. $\sqrt[3]{(-8a^6b^{15})}$	$-2a^2b^5$.
2. $\sqrt{(9x^2b^4)}$	$\pm 3xb^2$.
3. $\sqrt{(x^{2n})}$	$\pm x^n$.
4. $\sqrt[n]{(x^{2n})}$	x^2.
5. $(16a^{16})^{\frac{1}{4}}$	$\pm 2a^4$.
6. $(x^{\frac{1}{2}})^{\frac{1}{6}}$	$x^{\frac{1}{12}}$.
7. $\sqrt[3]{(-125a^3b^6)}$	$-5ab^2$.
8. $\sqrt{\frac{4}{9}\frac{a^3}{b^4}}$	$\pm\frac{2}{3}\cdot\frac{a^{\frac{3}{2}}}{b^2}$.
9. $(x^{-\frac{1}{3}})^{-\frac{1}{4}}$	$x^{\frac{1}{12}}$.
10. $\sqrt{(\sqrt{y})}$	$\pm\sqrt[4]{y}$.
11. $\left(\frac{32x^{10}y^5}{243b^{20}a^{15}}\right)^{\frac{1}{5}}$	$\frac{2x^2y}{3a^3b^4}$.
12. $\sqrt[b]{x^a}$	$x^{\frac{a}{b}}$.
13. $\sqrt[m]{(a^{-m})}$	$\frac{1}{a^m}$.
14. $\sqrt{(5a^4b^{-6}x^{4n})}$	$\pm a^2b^{-3}x^{2n}\sqrt{5}$.
15. $(-27a^2b^{\frac{1}{2}}x^{-3})^{\frac{1}{3}}$	$-3a^{\frac{2}{3}}b^{\frac{1}{6}}x^{-1}$.
16. $\left(\frac{a^{-1}}{b^{3n}x^{\frac{1}{n}}}\right)^{\frac{1}{3}}$	$a^{-\frac{1}{3}}b^{-n}x^{-\frac{1}{3n}}$.
17. $\sqrt{\sqrt[3]{\sqrt[4]{a}}}$	$a^{\frac{1}{24}} = \sqrt[24]{a}$.

Quantities to be evolved.	ANSWERS.
18. $\left(\frac{3^{10}a^{-5}(a+b)^5(2+x)^{-10}}{32c^5d^{-15}}\right)^{\frac{1}{5}}$	$\frac{9d^3(a+b)}{2ac(2+x)^2}$.
19. $[(a^{-1})^{-2}]^{-\frac{1}{2}}$	$\frac{1}{a^{\frac{1}{2}}}$.
20. $\sqrt[3]{\frac{27x^{12}(a+x)^{-6n}}{8b^{-3n}c^9}}$	$\frac{3b^nx^4}{2c^3(a+x)^{2n}}$.
21. $(x^{16})^{\frac{1}{4}}$.	
22. $(x^{30})^{\frac{1}{10}}$.	
23. $(x^{2n+4})^{\frac{1}{2}}$.	
24. $\sqrt{(a+x)^6}$.	
25. $(9a^4x^2y^{-12})^{\frac{1}{2}}$.	
26. $(169x^{2m+2}a^{-2n})^{\frac{1}{2}}$.	
27. $\sqrt[3]{(a^2+x^2)^{\frac{1}{2}}}$.	
28. $\sqrt[3]{\frac{64a^9x^{12}y^{-15}}{125x^6(a+x)^3}}$.	
29. $\sqrt[3]{-\frac{343a^{12}x^{-3}y^4}{729b^9c^{15}z}}$.	
30. $\left(\frac{32a^{10}y^{15}}{6^5}\right)^{\frac{1}{5}}$.	
31. $\left[\frac{}{(x^{\frac{1}{2}})^{\frac{1}{3}}}^{\frac{1}{8}}\right]^{\frac{1}{4}}$	
32. $\left(\frac{1024b^{100}x^5z^{10}}{10{,}000{,}000{,}000a^{20}}\right)^{\frac{1}{10}}$.	
33. $\sqrt[3]{[(a^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}}$.	

97. II. *To extract the square root of a compound quantity.* (1) Arrange the terms of the compound quantity according to the dimensions of some one letter, placing the highest power first, the next highest second, and so on; (2) find (Art. 93) the

root of the first term, set it down as the first term of the required root, and subtract its square from the given quantity; (3) take the quotient of the first term of the remainder by double the portion of the root already found, and set it down as the next term of the root, annexing it, at the same time, to the double of the part of the root already found; then multiply by it the compound quantity thus formed, subtract the product from the former remainder, and repeat the process as often as may be required.

Thus, to extract the square root of $2ab+b^2+a$; arranging (1) the terms according to the powers of a, we have (2) a, the square root of the first term, for the first part of the root; subtracting its square, the remainder is $2ab+b^2$, the quotient of the first term of which, by the double of the root (3), is connected both to the divisor and root; then subtracting the product of the new divisor by this root, there is no remainder, and the root is complete.

$$\begin{array}{l|l} a^2+2ab+b^2 & a+b \\ \hline a^2 & \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & 0 \end{array}$$

Again, to find the square root of $a^2+2ab+b^2+2c(a+b)+c^2$, we determine the first and second terms of the root as before; we then repeat the same process upon the remainder, dividing by the double of $a+b$, as we before divided by the

$$\begin{array}{l|l} a^2+2ab+b^2+2c(a+b)+c^2 & a+b+c \\ \hline a^2 & \\ \hline 2a+b & 2ab+b^2+2c(a+b)+c^2 \\ & 2ab+b^2 \\ \hline & 2(a+b)+c \quad 2c(a+b)+c^2 \\ & 2c(a+b)+c^2 \\ \hline & 0 \end{array}$$

double of a; we then find c for the quotient, and when it is connected to the root and divisor, and the product taken, there is no remainder.

If the given square be not arranged according to the powers of one letter, a root could not be found in finite terms. *See Note*, p. 46.

The reason of the rule may be explained as follows: Since Evolution consists in reversing the process of Involution, it is only by observing what law the formation of the square follows, that we can arrive at a method for discovering the square root. Now, the square of a binomial consists of the squares of the two terms, and twice their product: then, in the first example, in which it is required to extract the square root of $a^2+2ab+b^2$, it is plain that the first term of the root must be a. Again, the second term of the square is twice the product of the two terms of the root; therefore if this second term, $2ab$, be divided by the double of the first term of the root, that is, by $2a$, the quotient will be the second term of the root. Then, in order to determine whether the two last terms of the given square will be exactly reproduced, so as to leave no remainder, the second part, b, of the root must be joined to the divisor, $2a$; and thus, by multiplication and subtraction, the two last terms, namely, $2ab+b^2$, are reproduced.

In Example 3, Art. 92, we have seen how a trinomial may be expressed under the form of a binomial, by considering two of the terms as one; and hence the law of the formation of the square of a binomial applies to that of a trinomial also; and the same law holds respecting the square of a quantity consisting of any number of terms. It is upon this principle that the root has been found in the second Example of the Rule.

When we apply the Rule to quantities which are not complete squares, that is, to quantities which are not composed of two equal factors, the attempt to find a finite quotient fails, and an infinite series is obtained; in the same manner as the application of the Rule of Division to quantities which have no factors, leads to an interminable quotient.

EXAMPLES.

1. Extract the square root of $x^6+4x^5+2x^4+9x^2-4x+4$.

$$
\begin{array}{l}
x^6+4x^5+2x^4+9x^2-4x+4 \,(x^3+2x^2-x+2 \\
\quad x^6 \\
\hline
2x^3+2x^2)\ 4x^5+2x^4 \\
\qquad 4x^5+4x^4 \\
\hline
2x^3+4x^2-x)\ -2x^4+9x^2-4x \\
\qquad\quad -2x^4-4x^3+x^2 \\
\hline
2x^3+4x^2-2x+2)\ 4x^3+8x^2-4x+4 \\
\qquad\qquad\quad 4x^3+8x^2-4x+4 \\
\hline
\qquad\qquad\qquad 0
\end{array}
$$

2. Find the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}-2$.

Here, since $\frac{b^2}{a^2}=b^2a^{-2}$, and $2=2a^0$, 2 is a higher power of a than $\frac{b^2}{a^2}$; the arrangement is therefore as under:

$$
\begin{array}{l}
\frac{a^2}{b^2}-2+\frac{b^2}{a^2}\,\left(\frac{a}{b}-\frac{b}{a}\right. \\
\quad \frac{a^2}{b^2} \\
\hline
\frac{2a}{b}-\frac{b}{a})\ -2+\frac{b^2}{a^2} \\
\qquad -2+\frac{b^2}{a^2} \\
\hline
\qquad 0
\end{array}
$$

3. Required the square root of $1+x$.

$$1+x(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}\ \&c.$$

$$\begin{array}{l|l}
 & 1 \\
\hline
2+\frac{x}{2} & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16} & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}\ \&c.
\end{array}$$

If the numerators and denominators of the two last terms of this root be multiplied by 3, the series may be put under the form

$$1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\frac{3.5.7x^5}{2.4.6.8.10}\ \&c.$$

from which the law of continuation is more obvious.

4. Extract the square root of x^2-6x+9.

Ans. $x-3$.

5. Extract the square root of $25a^2+60a+36$.

Ans. $5a+6$.

6. Extract the square root of $9x^2+24xy+16y^2$.

Ans. $3x+4y$.

7. Extract the square root of $a^2-ax+\frac{x^2}{4}$.

Ans. $a-\frac{x}{2}$.

8. Extract the square root of $2+x+\frac{1}{x}$.

Ans. $\sqrt{x}+\sqrt{\frac{1}{x}}$.

9. Extract the square root of $4x^4-16x^3+24x^2-16x+4$. *Ans.* $2x^2-4x+2$.

10. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$. *Ans.* $2a^2+3ax+x^2$.

11. Extract the square root of a^2+x^2.

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$ &c.

98. III. *To extract the cube root of a compound quantity.* (1) Arrange the terms according to the dimensions of some one letter; (2) find the cube root of the first term, and subtract its cube from the given quantity; (3) take the quotient of the first term of the remainder by three times the square of the part of the root already found, and set it down as the next term of the root: then, to the treble of the square of the first term of the root, add three times the product of the two terms of the root, together with the square of the last term; multiply the compound quantity thus formed by the term of the root last found, subtract the product from the former remainder, and repeat the process as often as may be required.

The second term of the cube of a binomial, is always the product of the second term of the root, by three times the square of the first: and $a^3+3ab^2+3a^2b+b^3$, the general expression for the cube of a binomial, may be put under the form

$$a^3+(3a^2+3ab+b^2)b.$$

The above rule is nothing else than these results, ex-

pressed in the form of directions for an operation. The principle is extended to trinomials, and other polynomials, in the same manner as in the case of the square root.

EXAMPLES.

1.

$$a^3+3a^2b+3ab^2+b^3(a+b$$
$$a^3$$
$$3a^2+3ab+b^2)3a^2b+3ab^2+b^3$$
$$3a^2b+3ab^2+b^3$$
$$0$$

2.

$$(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3(a+b+c$$
$$(a+b)^3$$
$$3(a+b)^2+3(a+b)c+c^2)3(a+b)^2c+3(a+b)c^2+c^3$$
$$3(a+b)^2c+3(a+b)c^2+c^3$$
$$0$$

3.

$$27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$$
$$27x^6 \qquad (3x^2-2x+1$$
$$27x^4-18x^3+4x^2)-54x^5+63x^4-44x^3$$
$$-54x^5+36x^4-8x^3$$
$$27x^4-36x^3+21x^2-6x+1)27x^4-36x^3+21x^2-6x+1$$
$$27x^4-36x^3+21x^2-6x+1$$
$$0$$

4. Find the cube root of $x^3+6x^2+12x+8$.

Ans. $x+2$.

5. Find the cube root of $x^3+3x^2y+3xy^2+y^3$.

Ans. $x+y$.

6. Find the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Ans.* x^2-2x+1.

7. Find the cube root of $8x^3-60x^2+150x+125$.

Ans. $2x-5$.

99. Since (Art. 86) any root may be expressed as a power with a fractional index, we may extend the application of the binomial theorem to the extraction of roots, by taking for the index n, the fraction which expresses the root to be extracted: and it might be shown that in this case, as in Example 2, Art. 92, the series will never terminate.

EXAMPLES.

1. Expand $\sqrt{(a^2+x^2)}$ by the binomial theorem.

Here, since $\sqrt{(a^2+x^2)} = (a^2+x^2)^{\frac{1}{2}}$, we must put for a, b, and n, in the general formula, a^2, x^2, and $\frac{1}{2}$: then

$$(a^2+x^2)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} + \tfrac{1}{2}(a^2)^{-\frac{1}{2}}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2}(a^2)^{-\frac{3}{2}}x^4 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{2 \cdot 3}(a^2)^{-\frac{5}{2}}x^6 \text{ \&c.}$$

or,
$$\sqrt{(a^2+x^2)} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} \text{ \&c.}$$

This is the same series which is found by actual evolution, in Exercise 11, Prob. II.

2. Find the value of $\sqrt[3]{9}$ in a series.

Here we have $\sqrt[3]{9} = (8+1)^{\frac{2}{3}}$; then

$$(8+1)^{\frac{2}{3}} = 8^{\frac{2}{3}} + \tfrac{2}{3}.8^{-\frac{1}{3}} + \frac{\frac{2}{3}(-\frac{1}{3})}{2}8^{-\frac{4}{3}} + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{2 \cdot 3}8^{-\frac{7}{3}} + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{2 \cdot 3 \cdot 4}8^{-\frac{10}{3}} \text{ \&c.;}$$

or,

$$\sqrt[3]{9} = 2 + \tfrac{2}{3}\frac{1}{(8^{\frac{1}{3}})^2} - \frac{2}{3.3.2}\cdot\frac{1}{(8^{\frac{1}{3}})^5} + \frac{2.5}{3.3.3.6}\cdot\frac{1}{(8^{\frac{1}{3}})^8} - \frac{2.5.8}{3.3.3.3.6.4}\cdot\frac{1}{(8^{\frac{1}{3}})^{11}} \text{ \&c.}$$

whence

$$\sqrt[3]{9} = 2 + \frac{1}{3.2^2} - \frac{1}{3.6.2^4} + \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} \text{ \&c.}$$

3. Find the value of $\sqrt{(b^2+x)}$ in a series.

Ans. $b+\frac{x}{2b}-\frac{x^2}{2.4b^3}+\frac{3x^3}{2.4.6b^5}-\frac{3.5x^4}{2.4.6.8b^7}$ &c.

4. Find the value of $\frac{1}{\sqrt{(b^2+x)}}$ in a series.

Ans. $\frac{1}{b}-\frac{x}{2.b^3}+\frac{3x^2}{2.4b^5}-\frac{3.5x^3}{2.4.6b^7}+\frac{3.5.7x^4}{2.4.6.8b^9}$ &c.

5. Find the value of $\sqrt{2}$ in a series.

Ans. $1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}$ &c.

6. Expand $\sqrt{(1+x)}$ into an infinite series.

Ans. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}$ &c.

7. Expand $\sqrt[n]{(1+x)}$ into an infinite series.

Ans. $1+\frac{1}{n}x-\frac{n-1}{2n^2}x^2+\frac{(n-1)(2n-1)}{2\ .\ 3n^3}x^3$ &c.

8. Find the value of $\sqrt[3]{(a+x)^2}$ in a series.

Ans. $a^{\frac{2}{3}}(1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2.a^3}-\frac{4.7x^4}{9^2.12a^4}+\frac{4.7.10x^5}{9^2.12.15a^5})$ &c.

9. Find the value of $\sqrt[3]{7}$ or $\sqrt[3]{(8-1)}$ in a series.

Ans. $2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}$ &c.

10. Find the value of $\frac{a}{\sqrt{(c^2+x^2)}}$ in a series.

Ans. $\frac{a}{c}(1-\frac{x^2}{2c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6})$ &c.

11. Expand $(a^2-x^2)^{-\frac{1}{2}}$ into a series.

Ans. $\frac{1}{a}+\frac{1}{2}.\frac{x^2}{a^3}+\frac{3}{8}.\frac{x^4}{a^5}+\frac{5}{16}.\frac{x^6}{a^7}$ &c.

12. Expand $(1+x)^{-\frac{1}{n}}$ into a series.

Ans. $1-\frac{1}{n}x+\frac{n+1}{2n^2}x^2-\frac{(n+1)(2n+1)}{2\ .\ 3\ .\ n^3}$ &c.

CHAP. V.—SURDS.

FUNDAMENTAL PRINCIPLES.

ART. 100. Any root of a quantity which cannot be exactly evolved, is called a *surd*, or *radical*, or an *irrational quantity*: all other quantities are called *rational*. Thus, $\sqrt{8}$, $\sqrt[3]{9}$, are irrational quantities; but $\sqrt[3]{8}$, $\sqrt{9}$, are rational: also, $\sqrt[3]{(2a)}$, $\sqrt{(a^2+x^2)}$, $\sqrt[3]{6}$, $\sqrt[5]{(10b^2)}$, &c. are surds. Surds are distinguished from one another by the degree of the root; thus $\sqrt{a}$, $\sqrt{8}$, are *quadratic* surds, $\sqrt[3]{9}$, $\sqrt[3]{2a}$ are *cubic* surds, &c.

101. PROP. I. The square root of a quantity cannot be partly rational and partly irrational.

For, if possible, let $\sqrt{a} = b + \sqrt{c}$; then (Art. 31) $a = b^2 + 2b\sqrt{c} + c$, and (Art. 28) $a - b^2 - c = 2b\sqrt{c}$: hence (Art. 30), $\frac{a-b^2-c}{2b} = \sqrt{c}$, a rational quantity equal to an irrational, which is impossible.

102. II. In any expression of the form $a+\sqrt{b} = c+\sqrt{d}$, the rational parts are equal to one another, and also the irrational parts.

For, if a be not equal to c, let $a = c+x$, then $c+x+\sqrt{b} = c+\sqrt{d}$; therefore (Art. 28) $x+\sqrt{b} = \sqrt{d}$, which is impossible (Art. 101).

103. III. If $\sqrt{(a+\sqrt{b})} = c+\sqrt{d}$, then will $\sqrt{(a-\sqrt{b})} = c-\sqrt{d}$.

Since (Art. 31) $a+\sqrt{b} = c^2+2c\sqrt{d}+d$, we have

(Art. 102) $a = c^2+d$, and $\sqrt{b} = 2c\sqrt{d}$; whence (Art. 28) $a-\sqrt{b} = c^2-2c\sqrt{d}+d$, and (Art. 31, 97), $\sqrt{(a-\sqrt{b})} = c-\sqrt{d}$.

104. IV. If the product of two quadratic surds is a rational quantity, one of the surds must be a rational multiple of the other.

Let $\sqrt{x}\times\sqrt{y} = m$, a rational quantity; then $\sqrt{y} = \frac{m}{\sqrt{x}} = \frac{m}{x}\sqrt{x}$; and since m and x are rational, $\sqrt{y}$ is a rational multiple of $\sqrt{x}$. Hence, if one quadratic surd is not a rational multiple of another, their product cannot be a rational quantity.

105. V. One quadratic surd cannot be made up of two others, not having the same irrational part.

For, if possible, let $\sqrt{x} = \sqrt{m}+\sqrt{n}$; then $x = m+n+2\sqrt{mn}$, and $\frac{x-m-n}{2} = \sqrt{mn}$, a rational quantity equal to an irrational, which is absurd.

106. VI. The difference of any two equal powers is always divisible by the difference of their roots; that is, x^n-y^n is divisible by $x-y$.

Let $x-y = d$, therefore (Art. 27) $x = d+y$; then $x^n-y^n = (d+y)^n-y^n$. Expanding $(d+y)^n$ by the binomial theorem, we have

$$(d+y)^n = d^n+nd^{n-1}y+n\frac{n-1}{2}d^{n-2}y^2+ \text{ \&c. } \ldots + ndy^{n-1}+y^n;$$

therefore (Art. 28),

$$(d+y)^n-y^n = d^n+nd^{n-1}y+n\frac{n-1}{2}d^{n-2}y^2+ \text{ \&c. } \ldots\ldots + ndy^{n-1}.$$

Now, d will obviously divide all the terms of this latter part; therefore, $(d+y)^n - y^n$, or $x^n - y^n$, is also divisible by d, that is, by $x-y$. *See Note*, p. 56.

107. VII. The difference of two equal powers is always divisible by the sum of the roots, when the exponent is even.

Let $x+y=s$, therefore (Art. 28,) $x=s-y$; then $x^n - y^n = (s-y)^n - y^n$; and if $(s-y)^n$ be expanded, its last term, y^n, will be positive (Art. 92, and hyp.): then subtracting y^n as before, the remaining terms will be divisible by s; therefore $(s-y)^n - y^n$, or $x^n - y^n$, is also divisible by s, that is, by $x+y$.

108. VIII. The sum of two equal powers is always divisible by the sum of the roots, when the index is odd.

Let $x+y=s$, therefore $x=s-y$, and $x^n+y^n=(s-y)^n+y^n$. In this the last term of the expansion will be negative: hence, if y^n be added (Art. 27), the terms of the expansion will be divisible by s: $(s-y)^n+y^n$, or x^n+y^n, will also be divisible by s, or $x+y$.

109. PROB. I. *To reduce a quantity to the form of a given surd.* Raise the quantity to the power expressed by the index of the given surd, and indicate the extraction of the same root.

EXAMPLES.

1. Reduce $4a^{\frac{1}{2}}$ to the form of the cube root.

Cubing $4a^{\frac{1}{2}}$, we have $64a^{\frac{3}{2}}$, and the required form therefore is $(64a^{\frac{3}{2}})^{\frac{1}{3}}$.

2. Express $2a\frac{b^{\frac{1}{2}}}{3x^{\frac{1}{3}}}$ under the form of the fourth root.

Here raising to the fourth power, we have $\frac{16a^4b^2}{81x^{\frac{4}{3}}}$; the form required is $\left(\frac{16a^4b^2}{81x^{\frac{4}{3}}}\right)^{\frac{1}{4}}$.

3. Reduce $4\sqrt{5}$ to the form of the square root.

Here, $4\sqrt{5} = \sqrt{16} \times \sqrt{5} = \sqrt{80}$ (Art. 95.)

EXERCISES.

1. Reduce $2a$ to the form of the square root.
2. ..... $3x^2$ ..... square root.
3. ..... $2a^2b$ ..... cube root.
4. ..... $-3a$ ..... cube root.
5. ..... $3a^{\frac{1}{2}}$ ..... square root.
6. ..... b ..... nth root.
7. ..... $3\frac{a^2}{b^3}$ ..... 4th root.
8. ..... $3x^{\frac{1}{3}}$ ..... cube root.
9. ..... $-2n^2$ ..... 5th root.
10. ..... $ay^{\frac{1}{5}}$ ..... 4th root.
11. ..... $\frac{4}{5} \cdot \frac{b}{a^{\frac{1}{2}}}$ ..... square root.
12. ..... $\frac{1}{3}(3a)^{\frac{1}{2}}$ ..... square root.
13. ..... $\frac{2a}{x}(2b)^{\frac{1}{2}}$ ..... square root.
14. ..... $a-b$ ..... square root.
15. ..... $(x+y)^{\frac{1}{2}}$ ..... 4th root.
16. ..... $\frac{1}{4}\sqrt{8x}$ ..... square root.
17. ..... $2\sqrt[3]{3}$ ..... square root.

110. II. *To reduce a surd to its simplest form.* Resolve, if possible, the given surd into two factors, one of which shall be the greatest power contained in it, corresponding to the index of the surd, and

take the root of each factor. If the given surd be in a fractional form, multiply both terms of the fraction by such a number as will make the denominator a complete power, corresponding to the root, and proceed as before.

This process is merely the reverse of that in the last problem. It is plain from this mode of resolving into factors, that the surd will be in its simplest form.

EXAMPLES.

1. Reduce $x(a^n b)^{\frac{1}{n}}$ to its simplest form.

Here, $x(a^n b)^{\frac{1}{n}} = x \times ab^{\frac{1}{n}} = xab^{\frac{1}{n}}$.

2. Reduce $135^{\frac{1}{3}}$ to its simplest form.

Resolving 135, we have $135^{\frac{1}{3}} = (27 \times 5)^{\frac{1}{3}} = 27^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ (Art. 95) $= 3 \times 5^{\frac{1}{3}} = 3\sqrt[3]{5}$.

3. Reduce $2\sqrt[3]{\frac{8}{5}}$ to its simplest form.

Multiplying both terms so as to make the denominator a complete cube, we have $2\sqrt[3]{\frac{18}{27}} = 2\sqrt[3]{(\frac{1}{27} \times 18)} = \frac{2}{3}\sqrt[3]{18}$.

4. Reduce $(4a^5b^2 - 20a^3b^3 + 25ab^4)^{\frac{1}{2}}$ to its simplest form.

Here, $(4a^5b^2 - 20a^3b^3 + 25ab^4)^{\frac{1}{2}} = [(2a^2 - 5b)^2 \times ab^2]^{\frac{1}{2}} = (2a^2 - 5b)ba^{\frac{1}{2}}$.

EXERCISES.

Surds to be reduced.	ANSWERS.
1. $(9a^4x)^{\frac{1}{2}}$	$3a^2x^{\frac{1}{2}}$.
2. $\frac{3}{4}\sqrt{\frac{1}{2}}$	$\frac{3}{8}\sqrt{2}$.
3. $4\sqrt[3]{(8a^5 + 16a^3)}$	$8a(a^2+2)^{\frac{1}{3}}$.
4. $\sqrt{\frac{3}{4}}$	$\frac{1}{2}\sqrt{3}$.
5. $\frac{1}{3}\sqrt{\frac{5}{9}}$	$\frac{1}{9}\sqrt{5}$.
6. $\sqrt{\frac{4}{3}} = 2\sqrt{\frac{1}{3}}$	$\frac{2}{3}\sqrt{3}$.
7. $\left(\frac{a^3x - 4a^2x^2 + 4ax^3}{n^2p^2}\right)^{\frac{1}{2}}$	$\frac{a-2x}{np}(ax)^{\frac{1}{2}}$.

8. $(72x+108y)^{\frac{1}{2}}$ ..... $6\sqrt{(2x+3y)}$.

9. $\left(\frac{a^3b^2}{cd^2}-\frac{a^2b^3}{c^2d}\right)^{\frac{1}{2}}$ ..... $\frac{ab}{cd}(ac-bd)^{\frac{1}{2}}$.

10. $\frac{2x}{3a}(108x^{3n})^{\frac{1}{3}}$ ..... $\frac{2x^{n+1}}{a}\sqrt[3]{4}$.

11. $(3a^2c+6abc+3b^2c)^{\frac{1}{2}}$ ..... $(a+b)\sqrt{3c}$.

12. $(2ny^2-4ny+2n)^{\frac{1}{2}}$ ..... $(y-1)\sqrt{2n}$.

13. $(a^2b)^{\frac{1}{2}}$.

14. $(8a)^{\frac{1}{3}}$.

15. $75^{\frac{1}{2}}$.

16. $405^{\frac{1}{2}}$.

17. $147^{\frac{1}{2}}$.

18. $\sqrt[3]{162}$.

19. $(a^3+a^3b^2)^{\frac{1}{2}}$.

20. $3\sqrt{125}$

21. $(2x^3y^2+bx^5y)^{\frac{1}{2}}$.

22. $\frac{a}{b}(72n^8)^{\frac{1}{2}}$.

23. $(98a^2x)^{\frac{1}{2}}$.

24. $\frac{1}{2}\sqrt{\frac{5}{16}}$.

25. $\sqrt{\frac{2}{3}}$.

26. $\sqrt{\frac{a}{b}}$.

27. $\sqrt{\frac{2}{7}}$.

28. $\frac{1}{3}\sqrt[3]{\frac{4}{9}}$.

29. $\sqrt{\frac{1}{3}}$.

30. $\left(\frac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}\right)^{\frac{1}{2}}$.

31. $\frac{1}{3}\sqrt[3]{\frac{16}{81}}$.

111. III. *To reduce surds having different indices, to equivalent ones having a common index.* Reduce the indices to a common denominator, involve the

quantities to the power denoted by the numerator, and indicate the extraction of the root expressed by the denominator.

The reason of the rule is manifest from Art. 71, 86.

EXAMPLES.

1. Reduce $2^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to surds expressing the same root. Here, $2^{\frac{1}{2}} = 2^{\frac{3}{6}}$, and $4^{\frac{1}{3}} = 4^{\frac{2}{6}}$; but $2^3=8$, and $4^2=16$, therefore $2^{\frac{3}{6}} = 8^{\frac{1}{6}}$, and $4^{\frac{2}{6}} = 16^{\frac{1}{6}}$: hence, $2^{\frac{1}{2}} = 8^{\frac{1}{6}}$, and $4_{\frac{1}{3}} = 16^{\frac{1}{6}}$; so that $8^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$ are the surds required.

2. Reduce $ax^{\frac{1}{n}}$ and $by^{\frac{1}{m}}$ to surds of the same index.

Here, $ax^{\frac{1}{n}}$ and $by^{\frac{1}{m}}$ become, when the indices are reduced, $ax^{\frac{m}{mn}}$ and $by^{\frac{n}{mn}}$; that is, $a(x^m)^{\frac{1}{mn}}$ and $b(y^n)^{\frac{1}{mn}}$, or, $a\sqrt[mn]{x^m}$ and $b\sqrt[mn]{y^n}$.

EXERCISES.

Surds to be reduced.	ANSWERS.
1. $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$	$(a^3)^{\frac{1}{6}}$ and $(a^4)^{\frac{1}{6}}$.
2. $2^{\frac{1}{4}}$ and $3^{\frac{1}{5}}$	$32^{\frac{1}{20}}$ and $81^{\frac{1}{20}}$.
3. $2a^{\frac{2}{3}}$ and $3a^{\frac{3}{4}}$	$(a^8)^{\frac{1}{12}}$ and $(a^9)^{\frac{1}{12}}$.
4. a^2 and $b^{\frac{3}{2}}$	$\sqrt[3]{a^6}$ and $\sqrt[3]{b^2}$.
5. $5^{\frac{2}{3}}$ and $6^{\frac{3}{4}}$	$(5^{10})^{\frac{1}{15}}$ and $(6^{12})^{\frac{1}{15}}$.
6. $2\sqrt[3]{3}$ and $3\sqrt{2}$.	
7. $(a+x)^{\frac{1}{2}}$ and $(b-x)^{\frac{1}{3}}$.	
8. x^n and $y^{\frac{1}{n}}$.	
9. $\sqrt{\frac{1}{2}}$ and $\sqrt[5]{\frac{3}{4}}$.	

112. IV. *To add surds.* Reduce them, if necessary, to their simplest forms, and proceed as in adding rational quantities.

The reason of the rule is obvious from Art. 23, 32, 110.

EXERCISES.

Surds to be added.	Answers.
1. $\sqrt{45}+\sqrt{20}$	$5\sqrt{5}$.
2. $2\sqrt[3]{5}+5\sqrt{6}$	$2\sqrt[3]{5}+5\sqrt{6}$.
3. $\sqrt{\frac{9}{5}}+\sqrt{\frac{4}{27}}+\sqrt{\frac{9}{48}}$	$2\frac{5}{36}\sqrt{3}$.
4. $80^{\frac{1}{2}}+125^{\frac{1}{2}}$	$9\times5^{\frac{1}{2}}$.
5. $\sqrt[3]{48}+\sqrt[3]{162}$	$5\sqrt[3]{6}$.
6. $2\sqrt{48}+9\sqrt[3]{108}$	$8\sqrt{3}+27\sqrt[3]{4}$.
7. $3\times32^{\frac{1}{4}}+5\times2^{\frac{1}{4}}$	$11\times2^{\frac{1}{4}}$.
8. $(3a^2b)^{\frac{1}{2}}+(27a^2b)^{\frac{1}{2}}$	$4a(3b)^{\frac{1}{2}}$.
9. $\sqrt{(45c^3)}+\sqrt{(80c^2)}+\sqrt{(5a^2c)}$	$(a+7c)\sqrt{5c}$.
10. $(18a^5b^3)^{\frac{1}{2}}+(50a^3b^3)^{\frac{1}{2}}$	$(3a^2b+5ab)\sqrt{2ab}$.
11. $\frac{1}{2}\sqrt[3]{\frac{1}{4}}+\frac{1}{4}\sqrt[3]{\frac{1}{32}}$	$\frac{5}{16}\sqrt[3]{2}$.
12. $2\sqrt{\frac{5}{3}}+\frac{1}{6}\sqrt{60}+\sqrt{15}+\sqrt{\frac{5}{3}}$	$\frac{11}{3}\sqrt{15}$.
13. $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}+\sqrt[3]{(128a^6x^3)}$	*Ans.* $(28+4a^2x)\sqrt[3]{2}$.
14. $(16a^3b)^{\frac{1}{3}}+(4a^2b)^{\frac{1}{2}}+(54a^3b)^{\frac{1}{3}}+(a^2b)^{\frac{1}{2}}$.	*Ans.* $5a(2b)^{\frac{1}{3}}+3a(b)^{\frac{1}{2}}$.
15. $6(4a^2)^{\frac{1}{6}}+2(2a)^{\frac{1}{3}}+(8a^3)^{\frac{1}{6}}$	$9\sqrt[3]{2a}$.

113. V. *To subtract surds.* Reduce them, if necessary, to their simplest forms, and proceed as in subtracting rational quantities.

EXERCISES.

Surds to be subtracted.	Answers.
1. $\sqrt[3]{192}-\sqrt[3]{24}$	$2\sqrt[3]{3}$.
2. $(9a^4x)^{\frac{1}{2}}-(4a^4x)^{\frac{1}{2}}$	$a^2x^{\frac{1}{2}}$.
3. $\frac{1}{2}\sqrt[3]{81}-\frac{1}{4}\sqrt[3]{375}$	$\frac{1}{4}\sqrt[3]{3}$.
4. $b(8a^6b)^{\frac{1}{3}}-a(a^3b^4)^{\frac{1}{2}}$	$a^2b^{\frac{4}{3}}$.
5. $\frac{1}{3}(486)^{\frac{1}{5}}-\frac{1}{2}(64)^{\frac{1}{5}}$	$\frac{1}{2}\times2^{\frac{1}{5}}$
6. $512^{\frac{1}{3}}-54^{\frac{1}{3}}$	$2\times16^{\frac{1}{3}}-3\times2^{\frac{1}{3}}$
7. $\sqrt[4]{32}-2\sqrt[3]{40}$	$2\times2^{\frac{1}{4}}-4\times5^{\frac{1}{3}}$

Surds to be subtracted.	Answers.
8. $\sqrt[3]{500}-\sqrt[3]{108}$	$2\sqrt[3]{4}$.
9. $(7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2})-5\sqrt[3]{128}$	$8\sqrt[3]{2}$.
10. $\sqrt[3]{192}+\sqrt[3]{81}-\sqrt[3]{-512}$	$8+7\sqrt[3]{3}$.
11. $\overline{25a^2b}^{\frac{1}{2}}+\overline{144a^2b}^{\frac{1}{2}}-\overline{289a^2b}^{\frac{1}{2}}$	0.
12. $\sqrt{\frac{ab^3}{c^2}}+\frac{1}{2c}\sqrt{(a^3b-4a^2b^2+4ab^3)}$	$\frac{a}{2c}\sqrt{ab}$.

114. VI. *To multiply surds.* Reduce them, if necessary, to equivalent ones having a common index, and proceed as in multiplying rational quantities.

The reason of the rule is plain from Art. 95, 111.

EXAMPLES.

1. Multiply $2^{\frac{1}{3}}$ by $3^{\frac{1}{2}}$

By reduction (Art. 111), these become $2^{\frac{2}{6}}$, and $3^{\frac{3}{6}}$ that is, $4^{\frac{1}{6}}$ and $27^{\frac{1}{6}}$: hence the product is $(4\times27)^{\frac{1}{6}}=108^{\frac{1}{6}}$.

2. Multiply $\frac{1}{2}\sqrt[3]{6}$ by $\frac{2}{3}\sqrt[3]{18}$.

Here reduction is unnecessary, and the product is $\frac{1}{3}\sqrt[3]{108}=\sqrt[3]{4}$.

3. Multiply $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}$ by $(a+x)^{\frac{1}{2}}$.

$$x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}$$
$$(a+x)^{\frac{1}{2}}$$
$$(ax+x^2)^{\frac{1}{2}}+a+x.$$

EXERCISES.

Surds to be multiplied.	Answers.
1. $3\sqrt{b}\times5\sqrt{x}$	$15\sqrt{bx}$.
2. $4^{\frac{1}{10}}\times5^{\frac{1}{10}}$	$20^{\frac{1}{10}}$.
3. $2\sqrt[3]{3}\times4\sqrt[3]{72}$.	48.
4. $a^{\frac{1}{2}}\times a^{\frac{1}{3}}$	$a^{\frac{5}{6}}$.

Surds to be multiplied.	Answers.
5. $3b^{\frac{1}{3}} \times 4a^{\frac{1}{4}}$	$12(a^3b^4)^{\frac{1}{12}}$.
6. $\frac{1}{2}\sqrt[3]{\frac{2}{3}} \times \frac{1}{4}\sqrt[3]{\frac{5}{6}}$	$\frac{1}{8}\sqrt[3]{15}$.
7. $2^{\frac{1}{2}} \times 3^{\frac{1}{3}} \times 5^{\frac{1}{4}}$	$648000^{\frac{1}{12}}$.
8. $3\sqrt{(4+6\sqrt{2})} \times 5\sqrt{2}$	$30\sqrt{(2+3\sqrt{2})}$.
9. $(5\sqrt[3]{4}-2\sqrt[3]{16}) \times (2\sqrt[3]{2}-3\sqrt[3]{4})$.	*Ans.* $44-8\sqrt[3]{4}-30\sqrt[3]{2}$.
10. $(a+x)^{\frac{2}{3}} \times (a-x)^{\frac{2}{3}}$	$(a^2-x^2)^{\frac{2}{3}}$.
11. $(\sqrt{a}+\sqrt{b}+\sqrt{c}) \times (\sqrt{a}+\sqrt{b}-\sqrt{c}) \times (\sqrt{a}+\sqrt{c}-\sqrt{b}) \times (\sqrt{b}+\sqrt{c}-\sqrt{a})$.	*Ans.* $-a^2-b^2-c^2+2(ab+ac+bc)$.
12. $\frac{6^{\frac{1}{3}}}{4} \times 2\frac{9^{\frac{1}{3}}}{15}$	$\frac{1}{10}\sqrt[3]{2}$.
13. $5\sqrt{3} \times 7\sqrt{\frac{8}{3}} \times \sqrt{2}$	140.
14. $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5}$	$7\sqrt[3]{15}$.
15. $(a^{2m-n}b^{5m+1}c^{3p})^{\frac{1}{m}} \times (a^m b^{m-1} c^{m-3p})^{\frac{1}{m}}$	a^2b^6c.
16. $\left(x+\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}\right) \times \left(x+\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}\right)$	*Ans.* x^2+px-q.
17. $a^{\frac{1}{2}} \times x^{\frac{1}{3}}$.	
18. $4 \times 2\sqrt[6]{3} \times \sqrt[6]{72}$.	
19. $a^{\frac{1}{3}} \times ba^{\frac{1}{3}} \times ca^{\frac{1}{3}}$.	
20. $(3+\sqrt{5}) \times (2-\sqrt{5})$.	
21. $(7+2\sqrt{6}) \times (9-5\sqrt{6})$.	
22. $(4+2\sqrt{2}) \times (2-\sqrt{2})$.	
23. $\frac{1}{2}\sqrt[3]{18} \times 5\sqrt[3]{20}$.	
24. $(1+x)^{\frac{1}{2}} \times (1-x)^{\frac{1}{2}}$	
25. $x^{\frac{m}{n}} \times x^{\frac{p}{q}}$.	
26. $(a+b^{\frac{1}{2}})^{\frac{1}{2}} \times (c+d^{\frac{1}{2}})^{\frac{1}{2}}$.	
27. $(7\sqrt{6}+2\sqrt{3}) \times (\sqrt{5}+\sqrt{6})$.	
28. $\sqrt[3]{(a^2-x^2)^2} \times \sqrt[3]{(a^2-x^2)}$.	

Surds to be multiplied.

29. $(2\sqrt{8}+3\sqrt{5}-7\sqrt{2})\times(\sqrt{72}-5\sqrt{20}-2\sqrt{2})$.

30. $\sqrt[n]{(a+\sqrt{b})}\times\sqrt[n]{(a-\sqrt{b})}$.

115. VII. *To divide surds.* Reduce them, if necessary, to equivalent ones having a common index, and proceed as in dividing rational quantities.

Divide $bc\sqrt[n]{(ab)}$ by $b\sqrt[n]{a}$.

Here the quotient is $c\sqrt[n]{b}$; for (Art. 95) $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$, and therefore $\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}$.

EXERCISES.

Surds to be divided.	Answers.
1. $\sqrt{40}\div\sqrt{2}$	$\sqrt{20}=2\sqrt{5}$.
2. $6\sqrt{54}\div3\sqrt{2}$	$6\sqrt{3}$.
3. $\sqrt[3]{135}\div\sqrt[3]{5}$	3
4. $\sqrt[3]{4}\div\frac{1}{2}\sqrt[3]{6}$	$\frac{2}{3}\sqrt[3]{18}$
5. $72^{\frac{1}{3}}\div3^{\frac{1}{3}}$	$\sqrt{2}$.
6. $x^{\frac{4}{3}}\div x$	$x^{\frac{1}{3}}$.
7. $\sqrt{48}\div2$	$2\sqrt{3}$.
8. $(1-x^2)^{\frac{1}{2}}\div(1+x)^{\frac{1}{2}}$	$(1-x)^{\frac{1}{2}}$.
9. $(n^2-m^2)^{\frac{1}{2}}\div(n-m)$	$\sqrt{\frac{n+m}{n-m}}$.
10. $(ab^2-b^2x)^{\frac{1}{2}}\div(a-x)^{\frac{1}{2}}$	b.
11. $(\sqrt{72}+\sqrt{32}-4)\div\sqrt{8}$	$5-\sqrt{2}$.
12. $a^{\frac{1}{6}}\div a^{\frac{1}{3}}$	$a^{\frac{1}{6}}$.
13. $\left(a^3-a^2(a^3b^2)^{\frac{1}{3}}-2(a^2b^3)^{\frac{1}{2}}+2b^{\frac{1}{2}}.\right)\div(a^{\frac{1}{2}}-b^{\frac{1}{3}})$	*Ans.* $a^{\frac{5}{2}}-2b^{\frac{1}{2}}$.
14. $(a\sqrt{x}+\sqrt{bx}-a\sqrt{y}-\sqrt{by})\div(\sqrt{x}-\sqrt{y})$	$a+\sqrt{b}$.
15. $2\sqrt[3]{\frac{2}{3}}\div3\sqrt[3]{\frac{5}{4}}$.	
16. $5\frac{5}{4}\sqrt{\frac{1}{135}}\div\frac{2}{3}\sqrt{\frac{1}{3}}$.	

Surds to be divided.

17. $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$.

18. $64^{\frac{1}{3}} \div 2$.

19. $b(a^2 - x^2)^{\frac{1}{2}} \div (a+x)$.

20. $4\sqrt[3]{12} \div 2\sqrt{3}$.

21. $10\sqrt[3]{108} \div 5\sqrt[3]{4}$.

22. $a^{\frac{5}{6}} \div a^{\frac{1}{3}}$

23. $\left(\frac{a}{b}\right)^{\frac{1}{2}} \div \left(\frac{a}{b}\right)^{\frac{1}{2}}$.

24. $2\frac{1}{2}x^{\frac{1}{n}} \div 3\frac{1}{2}x^{\frac{1}{m}}$.

25. $x^{\frac{1}{2}} \div x^{\frac{1}{5}}$.

26. $10\sqrt{27} \div 2\sqrt{3}$.

27. $(\sqrt{6} + 4\sqrt{18} - 3 - 8\sqrt{2}) \div \sqrt{3}$.

28. $x^{\frac{2}{3}} \div x$.

29. $(a+x)^{\frac{3}{2}} \div (a+x)$.

30. $x^{\frac{m}{n}} \div x$.

31. $4 \div \sqrt{8}$.

116. VIII. *To extract the square root of a binomial surd.*

Let the binomial surd be $a \pm \sqrt{b}$, and let

$$\sqrt{(a \pm \sqrt{b})} = \sqrt{x} \pm \sqrt{y} :$$

hence, (Art. 31, 87), we have

$$a \pm \sqrt{b} = x \pm 2\sqrt{xy} + y.$$

Now, (Art. 102),

$$a = x + y, \ \sqrt{b} = 2\sqrt{xy} ;$$

therefore, (Art. 31, 87),

$$a^2 = x^2 + 2xy + y^2, \text{ and } b = 4xy :$$

hence (Art. 28),

$$a^2 - b = x^2 - 2xy + y^2$$

and (Art, 31, 97),

$$\sqrt{(a^2 - b)} = x - y ;$$

But

$$a=x+y;$$

therefore (Art. 27, 28),

$$a+\sqrt{(a^2-b)}=2x,\ a-\sqrt{(a^2-b)}=2y;$$

and (Art. 30),

$$\frac{a+\sqrt{(a^2-b)}}{2}=x,\ \frac{a-\sqrt{(a^2-b)}}{2}=y.$$

Hence (Art. 31)

$$\sqrt{x}\pm\sqrt{y} \text{ or } \sqrt{(a\pm\sqrt{b})}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}\pm\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}.$$

EXAMPLES.

1. Required the square root of $7-2\sqrt{10}$.

Assuming $\sqrt{x}-\sqrt{y}=\sqrt{(7-2\sqrt{10})}$, and following the steps of the preceding process, we have

$$x+y-2\sqrt{xy}=7-2\sqrt{10}$$

hence $x+y=7$, and $2\sqrt{xy}=2\sqrt{10}$

whence $x^2+2xy+y^2=49$

and $4xy=40$;

therefore $x^2-2xy+y^2=9$,

and $x-y=3$;

but $x+y=7$;

therefore $2x=10$, or $x=5$;

and $2y=4$, or $y=2$,

hence the required root is

$$\sqrt{x}-\sqrt{y}=\sqrt{5}-\sqrt{2}.$$

2. Find the square root of $4+2\sqrt{3}$.

Here, substituting, for a and $\sqrt{b}$ in the general formula, 4 and $2\sqrt{3}$, we have

$$\sqrt{(4+2\sqrt{3})}=\sqrt{\frac{4+\sqrt{(16-12)}}{2}}+\sqrt{\frac{4-\sqrt{(16-12)}}{2}}=\sqrt{3}+1.$$

3. Required the square root of $\sqrt{27}+\sqrt{24}$.

This may be worked by substitution, as in Ex. 2; if after the manner of Ex. 1 it may be first simplified, thus:

$$\sqrt{27}+\sqrt{24}=\sqrt{9\times3}+\sqrt{8\times3}=\sqrt{3}(3+\sqrt{8});$$

therefore $\sqrt{(\sqrt{27}+\sqrt{24})}=\sqrt[4]{3}\sqrt{(3+\sqrt{8})}$.

Now, $\sqrt{(3+\sqrt{8})}$ found as in Ex. 1. is $1+\sqrt{2}$; the root therefore is $\sqrt[4]{3}(1+\sqrt{2})=\sqrt[4]{3}+\sqrt[4]{12}$, since $\sqrt{2}=\sqrt[4]{4}$.

EXERCISES.

1. Find the square root of $7+4\sqrt{3}$. *Ans.* $2+\sqrt{3}$.
2. ..... $6-2\sqrt{5}$. *Ans.* $\sqrt{5}-1$.
3. ..... $7-2\sqrt{10}$. *Ans.* $\sqrt{5}-\sqrt{2}$.
4. ..... $2+\sqrt{3}$. *Ans.* $\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}$.
5. ..... $\frac{5}{2}+\sqrt{2}$. *Ans.* $1+\frac{1}{2}\sqrt{2}$.
6. ..... $23+8\sqrt{7}$. *Ans.* $4+\sqrt{7}$.
7. ..... $\sqrt{32}-\sqrt{24}$. *Ans.* $\sqrt[4]{18}-\sqrt[4]{2}$.
8. ..... $11+\sqrt{72}$ *Ans.* $3+\sqrt{2}$.
9. ..... $94-42\sqrt{5}$. *Ans.* $7-3\sqrt{5}$.
10. ..... $3\sqrt{5}+\sqrt{40}$. *Ans.* $\sqrt[4]{20}+\sqrt[4]{5}$.

117. IX. *To find such multipliers as will make binomial surds rational.* Performing the actual division which has been proved possible, in Art. 106, 107, 108, we have

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3 \text{ \&c.}$$

$$\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3 \text{ \&c.}$$

$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3 \text{ \&c.}$$

And it is plain that the first series will terminate at the nth term, whether n be even or odd; the second only when n is even, and the third only when n is odd.

Now, let $x^n=a$, $y^n=b$; therefore $x=\sqrt[n]{a}$, $y=\sqrt[n]{b}$,

$x^{n-1}=\sqrt[n]{a^{n-1}}$, $x^{n-2}=\sqrt[n]{a^{n-2}}$, &c.; and $y^2=\sqrt[n]{b^2}$, $y^3=\sqrt[n]{b^3}$, &c.

Substituting these values in the above series, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\sqrt[n]{a^{n-4}b^3}+$$
&c.

$$\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}$$
&c.

$$\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}$$
&c.

Hence, by the nature of division, if each of these irrational divisors be multiplied by the corresponding irrational quotient, the corresponding rational dividend will be reproduced; and, therefore, the first of these quotients is the multiplier, which will render rational a surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$; and the second and third quotients are the multipliers, which will make a surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ rational; the binomial which is produced having its second term *negative* or *positive*, according as n is even or odd.*

* In many cases, multipliers may be more readily found than by the above general method. Thus, it is plain that a surd of the form $\sqrt[n]{a}$ will be made rational by the multiplier $b\sqrt[n]{a^{n-1}}$; for $\sqrt[n]{a}\times\sqrt[n]{a^{n-1}}=\sqrt[n]{a^n}=a$. Also, since the product of the sum and difference of two quantities is the difference of their squares, (Note p. 35), it follows that $\sqrt{a}\pm\sqrt{b}$ will be made rational by the multiplier $\sqrt{a}\mp\sqrt{b}$. Again, since $(a\mp b)\times(a^2\pm ab+b^2)=a^3\mp b^3$, a surd of the form $\sqrt[3]{a}\mp\sqrt[3]{b}$ will be made rational by the multiplier $\sqrt[3]{a^2}\pm\sqrt[3]{ab}+\sqrt[3]{b^2}$. Several of the exercises in the text may be worked by this method.

EXAMPLES.

1. Required the multiplier which will make $\sqrt[3]{9}+\sqrt[3]{10}$ rational.

Here $a=9$, $b=10$, $n=3$; then, since n is odd, and the second term positive, the third form applies to this case; therefore we have $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}$ &c. $=\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}$. The actual multiplication is as under:

multiplier, $\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}$

given surd, $\sqrt[3]{9}+\sqrt[3]{10}$

$$9-\sqrt[3]{810}+\sqrt[3]{900}$$
$$\sqrt[3]{810}-\sqrt[3]{900}+10$$

rational product, $9+10=19$

The multiplier might have been more readily found, by applying the principle in the Note.

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction whose denominator shall be rational.

Here the multiplier is plainly $\sqrt{5}+\sqrt{2}$; then multiplying both terms by this, we have

$$\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\sqrt{5}+\sqrt{2}.$$

3. What multiplier will make $3^{\frac{1}{4}}-4^{\frac{1}{4}}$ rational.

The multiplier, found by the general formula, is $27^{\frac{1}{4}}+36^{\frac{1}{4}}+48^{\frac{1}{4}}+64^{\frac{1}{4}}$, but, by the principle in the Note, we may use the two multipliers, $3^{\frac{1}{4}}+4^{\frac{1}{4}}$ and $3^{\frac{1}{2}}+4^{\frac{1}{2}}$, in succession. In fact, the above multiplier is the product of these two.

EXERCISES.

1. Reduce $\frac{1}{\sqrt{3}+2}$ to a fraction with a rational denominator. *Ans.* $\frac{\sqrt{3}-2}{-1}=2-\sqrt{3}$.

2. Convert $a+\sqrt{b}$ into a rational quantity. *Ans.* a^2-b.

3. Convert $-5-\sqrt{\frac{3}{4}}$ into a rational quantity. *Ans.* $24\frac{1}{4}$.

4. Convert $\sqrt{a}+\sqrt{b}$ into a rational quantity. *Ans.* $a-b$.

5. Convert $a\sqrt{b}+c\sqrt{d}$ into a rational quantity. *Ans.* a^2b-c^2d.

6. Convert $9+2\sqrt{10}$ into a rational quantity. *Ans.* 41.

7. Reduce $\frac{1+\sqrt{2}}{2-\sqrt{2}}$ to a fraction with a rational denominator. *Ans.* $2+\frac{3}{2}\sqrt{2}$.

8. Reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ to a fraction with a rational denominator. *Ans.* $2\frac{8+2\sqrt{15}}{2}=8+2\sqrt{15}$.

9. Reduce $\frac{\sqrt{a}}{b+\sqrt{c}}$ to a fraction with a rational denominator. *Ans.* $\frac{\sqrt{a}(b-\sqrt{c})}{b^2-c}$.

10. Reduce $\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}$ to a fraction with a rational denominator. *Ans.* $\frac{1}{10}$.

11. Reduce $\sqrt[n]{\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}}$ to a fraction with a rational denominator. *Ans* $\sqrt[n]{\frac{2a+2\sqrt{ab}+b}{a-b}}$.

12. Reduce $\frac{3}{\sqrt[3]{5}-\sqrt[3]{2}}$ to a fraction with a rational denominator. *Ans.* $\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$.

13. Convert $\dfrac{\sqrt{(1-x)}+\dfrac{1}{\sqrt{(1+x)}}}{1+\dfrac{1}{\sqrt{(1-x^2)}}}$ into a quantity with a rational denominator. *Ans* $\sqrt{(1-x)}$.

14. Convert $\dfrac{\sqrt{(x+a)}+\sqrt{(x-a)}}{\sqrt{(x+a)}-\sqrt{(x-a)}}$ into a quantity with a rational denominator.

Ans. $\dfrac{x+\sqrt{(x^2-a^2)}}{a}$.

15. Show that $\dfrac{a\sqrt{(a+x)}}{\sqrt{(a+x)}-\sqrt{x}} = a+x+\sqrt{(ax+x^2)}$.

CHAP. VI.—IMAGINARY QUANTITIES.

ART. 118. The even roots of negative quantities, as has been already remarked in Art. 96, are not assignable by any process of Evolution; and, hence, expressions involving such roots have been called *imaginary* or *impossible* quantities. In certain cases, however, a real value can be assigned to them, but not by ordinary Evolution.

119. Since (Art. 95) the square root of a product equals the product of the square roots of its factors, and since $-a^2=a^2\times-1$, it follows that $\sqrt{-a^2}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1}$; so also $\sqrt{-b}=\sqrt{b}\times\sqrt{-1}$ and $\sqrt[4]{-a^4}=a\sqrt[4]{-1}$, &c.; and thus any imaginary quantity may be resolved into two factors, one a real quantity, and the other the imaginary expression $\sqrt{-1}$. Further, from the nature of powers and roots it is plain that the square of $\sqrt{-1}$ is -1, or that $\sqrt{-1}\times\sqrt{-1}=-1$. Attention to this principle, and to the above mode of notation, will render all the algebraic operations with imaginary quantities very easily performed.

Thus, the sum of $a+3b\sqrt{-1}$ and $a-2b\sqrt{-1}$ is $2a+b\sqrt{-1}$; and the product of $a\sqrt{-b}$ by $c\sqrt{-d}$ is $a\sqrt{b}\sqrt{-1}\times c\sqrt{d}\sqrt{-1}=ac\sqrt{bd}\times-1=-ac\sqrt{bd}$. Also to divide $a\sqrt{-1}$ by $b\sqrt{-1}$, we have $\frac{a\sqrt{-1}}{b\sqrt{-1}}=\frac{a}{b}$. These two latter results are real, the imaginary quantities having disappeared. The same occurs frequently in operating with these quantities; as in the following example, which is besides otherwise interesting.

Let it be required to multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$. The work is as follows:—

$$\begin{array}{r} a+b\sqrt{-1} \\ a-b\sqrt{-1} \\ \hline a^2+ab\sqrt{-1} \\ -ab\sqrt{-1}+b^2 \\ \hline a^2+b^2 \end{array}$$

Hence, conversely, it appears that we can resolve a binomial whose terms are positive into two compound factors, which are the sum and difference of a real and an imaginary quantity. Thus, since $16a^2+9x^2=16a^2-9x^2\times-1=9x^2-16a^2\times-1$; and since the square roots of the terms are $4a$ and $3x\sqrt{-1}$, the factors of the given binomial will be $4a+3x\sqrt{-1}$, and $4a-3x\sqrt{-1}$, or $3x+4a\sqrt{-1}$, and $3x-4a\sqrt{-1}$. *See Note p.* 35.

120. In order to determine the sign of the product of any number of imaginary quantities it is not sufficient that we know merely the signs which are prefixed to these quantities; it is essential that we know also how often $\sqrt{-1}$ is involved; and hence it is of importance to determine the successive powers of this expression. Now $(\sqrt{-1})^0=1$, (Art. 84), $(\sqrt{-1})^1=\sqrt{-1}$, $(\sqrt{-1})^2=-1$; therefore $(\sqrt{-1})^3=-1\times\sqrt{-1}=-\sqrt{-1}$, $(\sqrt{-1})^4=-1\times-1=1$; and in the same manner we have,

1. $(\sqrt{-1})^{4n}=[(\sqrt{-1})^4]^n=1^n=1$.
2. $(\sqrt{-1})^{4n+1}=(\sqrt{-1})^4\times\sqrt{-1}=\sqrt{-1}$.
3. $(\sqrt{-1})^{4n+2}=(\sqrt{-1})^{4n}\times(\sqrt{-1})^2=-1$.
4. $(\sqrt{-1})^{4n+3}=(\sqrt{-1})^{4n+2}\times\sqrt{-1}=-\sqrt{-1}$.

It is plain that all the powers of $\sqrt{-1}$ are comprehended in these four forms, since, n being a whole number or zero,

all positive integers are of one or other of the forms $4n$, $4n+1$, $4n+2$, $4n+3$. Thus

if $n=0$ these forms are the 0th, 1st, 2d, 3d, powers of $\sqrt{-1}$,
.. $n=1$ ..... 4th, 5th, 6th, 7th, ...
.. $n=2$ ..... 8th, 9th, 10th, 11th, ...
.. $n=3$ ..... 12th, 13th, 14th, 15th, &c. ....

Hence,

the 0th, 4th, 8th, 12th, $4n$th powers of $\sqrt{-1}$ are each 1,
.. 1st, 5th, 9th, 13th, $(4n+1)$th ..... $\sqrt{-1}$,
.. 2nd, 6th, 10th, 14th, $(4n+2)$th ..... -1,
.. 3rd, 7th, 11th, 15th, $(4n+3)$th ..... $-\sqrt{-1}$,

and so on of the others.*

EXERCISES.

1. Find the sum and difference of $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$. *Ans.* $2a$ and $2b\sqrt{-1}$.

2. Add together $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$.
Ans. $a+c+(b+d)\sqrt{-1}$.

3. From $4a+2b\sqrt{-1}$ subtract $3a+4b\sqrt{-1}$.
Ans. $a-2b\sqrt{-1}$.

4. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$. *Ans.* $-ab$.

5. Multiply $\sqrt{-x}$ by $\sqrt{-y}$. *Ans.* $-\sqrt{xy}$.

6. Multiply $a\sqrt{-1}$ by $-b\sqrt{-1}$. *Ans.* ab.

* The same may, perhaps, be more easily found by actual multiplication, after the first four powers are known. Thus, the fifth power is $\sqrt{-1}$, being the product of the 4th, 1, by the 1st, $\sqrt{-1}$; the 6th is -1, the product of the 4th, 1, by the 2nd, -1; also multiplying the 3rd power, $-\sqrt{-1}$, by the 4th power, 1, we find for the 7th $-\sqrt{-1}$; multiplying the 4th power by itself we find 1 for the 8th; the 5th power $\sqrt{-1}$, multiplied by the 4th produces $\sqrt{-1}$ for the 9th, and so on for the 10th, 11th, and all higher powers.

7. Find the 3rd and 4th powers of $a\sqrt{-1}$.

Ans. $-a^3\sqrt{-1}$ and a^4.

8. Resolve 2 or $1+1$ into imaginary factors.

Ans. $1+\sqrt{-1}$ and $1-\sqrt{-1}$.

9. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$. *Ans.* $-6\sqrt{6}$.

10. Find the 3rd power of $a+b\sqrt{-1}$.

Ans. $a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}$.

11. Multiply $a\pm b\sqrt{-1}$ by $c\pm d\sqrt{-1}$.

Ans. $ac-bd\pm(ad+bc)\sqrt{-1}$.

12. Resolve $25a^4+16b^6$ into imaginary factors. *See note p.* 35. *Ans.* $5a^2+4b^3\sqrt{-1}$ and $5a^2-4b^3\sqrt{-1}$, or $4b^3+5a^2\sqrt{-1}$ and $4b^3-5a^2\sqrt{-1}$.

13. Find the sum and difference of the fractions

$$\frac{1}{a-b\sqrt{-1}}, \frac{1}{a+b\sqrt{-1}};$$ and also of

$$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}, \frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}.$$

Ans. $\frac{2a}{a^2+b^2}$, $\frac{2b\sqrt{-1}}{a^2+b^2}$ and $\frac{2(a^2-b^2)}{a^2+b^2}$, $\frac{4ab\sqrt{-1}}{a^2+b^2}$.

14. Resolve $a+b$ into imaginary factors.

Ans. $\sqrt{a}+\sqrt{b}\sqrt{-1}$ and $\sqrt{a}-\sqrt{b}\sqrt{-1}$.

15. Find the cube of $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

Ans. 1.

16. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.

17. Simplify the fraction $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}$ *Ans.* $\sqrt{-1}$.

18. Resolve 100 into imaginary factors.

Ans. $6+8\sqrt{-1}$ and $6-8\sqrt{-1}$ or $8+6\sqrt{-1}$ and $8-6\sqrt{-1}$.

19. Find the continued product of $x+a$, $x+a\sqrt{-1}$, $x-a$, and $x-a\sqrt{-1}$. *Ans.* x^4-a^4.

20. Show that $\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = \frac{a^2-b^2+2ab\sqrt{-1}}{a^2+b^2}$.

21. Of what number are $24+7\sqrt{-1}$ and $24-7\sqrt{-1}$ the imaginary factors? *Ans.* 625.

22. Divide $3-\sqrt{-2}$ by $2+3\sqrt{-2}$. *Ans.* $-\frac{1}{2}\sqrt{-2}$.

23. Multiply $e^{a\sqrt{-1}}$ by $e^{b\sqrt{-1}}$. *Ans.* $e^{(a+b)\sqrt{-1}}$.

24. Raise $e^{a\sqrt{-1}}$ to the nth power. *Ans.* $e^{na\sqrt{-1}}$.

25. Of what number are $20+15\sqrt{-1}$ and $20-15\sqrt{-1}$ the imaginary factors? *Ans.* 625.

CHAP. VII.—SIMPLE EQUATIONS.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

ART. 121. DEF. I. A statement of equality between two expressions, is called an *equation;* and the expressions are called the *members* or *sides* of the equation: thus, $4b+d=2a-c$ is an equation, and its sides or members are $4b+d$ and $2a-c$.

122. *Schol.* The chief use of an equation is to enable us to express one quantity in terms of others. Hence, if one of the quantities in an equation be unknown, and the others known, by expressing the former in terms of the latter, its value is determined. It is customary to denote unknown quantities by the last letters of the alphabet; while known quantities, if not expressed in numbers, are denoted by the first letters of the alphabet.

123 II. A *simple* equation is one which contains the first power only of the unknown quantity.

124. PROP. I. Any term may be transposed from one side of an equation to the other, if its sign be changed at the same time.

Let $x+a=b$, then $x=b-a$; for this is nothing else than subtracting a from both sides (Art. 28).

Again, let $x-a=b$, then $x=b+a$; for this is nothing else than adding a to both sides (Art. 27).

125. II. An equation may be cleared of fractions,

by multiplying all the terms by the least common multiple of the denominators (Art. 63).*

For, by this means, each numerator is rendered divisible by the denominator, (Art. 46, 48, 54), and (Art. 29) the two sides are still equal to one another. (*See* Example 4, Art. 76). Thus, to free the equation, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=1$, from fractions, multiply each term by 12, the least common multiple of the denominators; then we have

$$6x+4x+3x=12,$$

in which the terms are integral.

126. III. An equation may be cleared of surds, by transposing (Art. 124) the terms, so that each surd, in succession, may occupy one side, and the remaining terms the other side, and then raising each side to a power corresponding to the index of the surd.

For (Art. 87, 31) the surds will be made to disappear successively, and each result will still be an equation. Thus, let

$$\sqrt[n]{(a+x)}+b=c;$$

then, by transposition (Art. 124,)

$$\sqrt[n]{(a+x)}=c-b;$$

and, by involution,

$$a+x=(c-b)^n,$$

which is free from irrational terms. If there are more surds than one, the process must be repeated.

* This may also be effected, but not so readily, by multiplying all the terms either by each denominator in succession, by the product of all the denominators, or by any common multiple of them.

127. PROB. *To solve a simple equation.* (1) Clear the equation of fractions and surds;* (2) transpose the known terms to one side, and the unknown terms to the other, incorporating like quantities; (3) and divide both members (Art. 30) by the coefficient of the unknown quantity.

EXAMPLES.

1. Given $4-9x=14-11x$ to find the value of x.

Transposing, (part 2 of the Rule, and Art. 124) we have

$$11x-9x=14-4;$$

and, collecting the terms,

$$2x=10;$$

therefore, by division (part 3),

$$x=5;$$

which is the required value.

To verify this value of x, put 5 for x in the given equation, and it becomes

$$4-45=14-55;$$

that is,

$$-41=-41;$$

or,

$$41=41,$$

by changing both signs (Art. 124), or dividing both sides by -1.

When the two sides thus become identical, the value found for the unknown is said to be that which *satisfies*

* That is, of course, when the unknown quantity is contained in the fraction or surd. Before commencing the solution, we may sometimes simplify an equation, by transposing and incorporating like terms, or by division (Art. 30.)

the equation, or for which the equation *holds true.**

2. Given, $3x^2+4x-\frac{x^2}{3}-46x+2x^2=0$, to find the value of x.

Here, simplifying before clearing of fractions, (Note to the Rule) we have

$$5x-\frac{x}{3}-42=0;$$

then multiplying both terms by 3,

$$15x-x-126=0, \text{ or } 14x=126,$$

and by division, $x=9$

3. Given, $\frac{2y+3}{2y-3}=\frac{11}{9}$, required y.

This equation may be easily solved, by taking the sum and difference of the terms, and dividing the former of these by the latter. We thus find $\frac{4y}{6}=\frac{20}{2}$,

or, $4y=60$, and $y=15$.

4. Given, $\frac{5x+4}{2}:\frac{18-x}{4}::7:4$, to find x.

Since, in a proportion, the product of the means is equal to the product of the extremes (Euc. VI. 16), it follows that

$$\frac{5x+4}{2}\times 4=\frac{18-x}{4}\times 7,$$

whence, we easily find

$$x=2.$$

* It is of great use to employ this test frequently, in the solution of equations. Thus, in the above example, if we had forgotten to change the sign of 4, in transposing it to the right side, we should have had $11x-9x=18$, or $2x=18$, and $x=9$; and, in attempting to verify this, we should have found $4-81=14-99$, or $77=85$, an impossibility; which proves that 9 is not the true value of x.

5. Given, $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$; required x.

Multiplying both sides by 18, we find

$$8x+6+\frac{126x-522}{5x-12}=8x+19;$$

therefore, cancelling $8x$, and transposing 6,

$$\frac{126x-522}{5x-12}=13.$$

Multiplying by the denominator, transposing and dividing, we find $x=6$.

6. Given, $\frac{x-3}{2\frac{1}{2}}-\frac{x-4}{6\frac{1}{3}}=3-\frac{x+1}{5}$; required the value of x.

Reducing $2\frac{1}{2}$ and $6\frac{1}{3}$ to improper fractions, and performing the division, we obtain

$$\frac{2(x-3)}{5}-\frac{3(x-4)}{19}=3-\frac{x+1}{5}.$$

Now, 95 is the least common multiple of the denominators; therefore,

$$38(x-3)-15(x-4)=285-19(x+1);$$

or actually multiplying,

$$38x-114-15x+60=285-19x-19.$$

Collecting the terms,

$$42x=320;$$

and

$$x=7\tfrac{26}{42}.$$

Verify this equation.

7. Given, $3.25x-5.007-x=0.2-0.34x$.

By transposition, we have

$$3.25x-x+0.34x=5.207;$$

that is,

$$x(3.25-1+0.34)=5.207;$$

whence,

$$2.59x = 5.207,$$

and

$$x = \frac{5.207}{2.59} = 2.0104, \text{ \&c.}$$

8. Given, $ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx+4a}{4}$.

Here, $4a$ is the least common multiple of the denominators: multiplying by it, we obtain

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2;$$

then, by transposition and addition,

$$x(4a^2 - 3ab) = 4a^2b^2 - 10a^2;$$

and by division,

$$x = \frac{4a^2b^2 - 10a^2}{4a^2 - 3ab} = \frac{4ab^2 - 10a}{4a - 3b},$$

by reduction to lowest terms.

9. Given, $\frac{ax - b^2}{\sqrt{ax}+b} = c + \frac{\sqrt{ax} - b}{c}$; required the value of x.

In this case, before squaring, the equation may be simplified. The numerator of the first term is divisible by the denominator. Performing the division, there is obtained

$$\sqrt{ax} - b = c + \frac{\sqrt{ax} - b}{c}.$$

By transposition and reduction to a common denominator this becomes

$$\frac{c(\sqrt{ax} - b)}{c} - \frac{\sqrt{ax} - b}{c} = c;$$

whence, separating the factors,

$$\frac{c-1}{c}(\sqrt{ax}-b)=c.$$

Dividing by the known coefficient, and then transposing b, we obtain

$$\sqrt{ax}=b+\frac{c^2}{c-1};$$

squaring both sides, and dividing by a, there results

$$x=\frac{1}{a}\cdot\left(b+\frac{c^2}{c-1}\right)^2.$$

10. Given, $\sqrt{a}+\sqrt{x}=\sqrt{\overline{ax}}$, to find the value of x.

Transposing $\sqrt{x}$, and resolving the right member into its factors, we have

$$\sqrt{a}=\sqrt{x}(\sqrt{a}-1);$$

by division,

$$\frac{\sqrt{a}}{\sqrt{a}-1}=\sqrt{x};$$

and by involution,

$$x=\frac{a}{(\sqrt{a}-1)^2}.$$

11. Given, $\dfrac{\sqrt{a}+\sqrt{(a-x)}}{\sqrt{a}-\sqrt{(a-x)}}=\dfrac{1}{a}$, required x.

Employing the same artifice as is used in Ex. 3, we shall have

$$\sqrt{\frac{a}{a-x}}=\frac{1+a}{1-a}.$$

Squaring this and then inverting the terms, we obtain

$$\frac{a-x}{a}=\frac{(1-a)^2}{(1+a)^2};$$

or,

$$1-\frac{x}{a}=\frac{(1-a)^2}{(1+a)^2};$$

by transposition,

$$\frac{x}{a} = 1 - \frac{1-2a+a^2}{1+2a+a^2} = \frac{4a}{(1+a)^2};$$

hence,

$$x = \frac{4a^2}{(1+a)^2} = \left(\frac{2a}{1+a}\right)^2.$$

The solution will not, in general, be simplified by the use of this artifice, unless when the unknown occurs only on one side.

12. Given, $\frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}}$ to find the value of x.

Squaring both sides, and cancelling equal terms, we find

$$\frac{1}{x^2} + \frac{2}{ax} = \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}.$$

Multiply the left member by x, and the right by its equal, $\sqrt{x^2}$, and there will be obtained

$$\frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{4}{a^2} + \frac{9}{x^2}\right)};$$

squaring,

$$\frac{1}{x^2} + \frac{4}{ax} + \frac{4}{a^2} = \frac{4}{a^2} + \frac{9}{x^2}.$$

Hence,

$$\frac{1}{a} = \frac{2}{x} \text{ and } x = 2a.$$

13. Given, $\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)} = b$; required x.

Since $(a+b)^3 = a^3+b^3+3ab(a+b)$ (Art. 87), we have in this case,

$$a+x+a-x+3\sqrt[3]{(a^2-x^2)}.(\sqrt[3]{a+x}+\sqrt[3]{a-x}) = b^3;$$

but

$$\sqrt[3]{a+x}+\sqrt[3]{a-x} = b,$$

therefore

$$2a+3b\sqrt[3]{(a^2-x^2)} = b^3,$$

and

$$3b\sqrt[3]{(a^2-x^2)} = b^3-2a\ ;$$

by division, involution, and transposition,

$$x^2 = a^2-\left(\frac{b^3-2a}{3b}\right)^3;$$

therefore

$$x = \sqrt{a^2-\left(\frac{b^3-2a}{3b}\right)^3}.$$

14. Let $(3+y)^{\frac{1}{n}} = (10+y^2)^{\frac{1}{2n}}$, to find y.

Raising both members to the $2n$th power will give

$$(3+y)^2 = 10+y^2\ ;$$

whence, $$y = \tfrac{1}{6}.$$

15. Given $(a^2+c)^{\frac{1}{x}} = \left(\frac{a^2+c}{d(x+b)}\right)^{\frac{1}{4}}$; required x.

Involving both members to the 4th power, and clearing of fractions, we find

$$(a^2+c)^{\frac{4}{3}} \cdot d(x+b) = a^2+c.$$

Dividing both terms by a^2+c, that is, by $\sqrt[3]{(a^2+c)^3}$, we find

$$\sqrt[3]{(a^2+c)} \cdot d(x+b) = 1\ :$$

hence,

$$x = \frac{1}{d\sqrt[3]{(a^2+c)}} - b.$$

EXERCISES.

I. *Numerical Equations.*

Given Equations.	Answers.
1. $x+4=7$	$x=3$.
2. $x-6=1$	$x=7$.
3. $2+x-3+7=10-3$	$x=1$.
4. $x-2=\pm 8$	$x=10$ or -6.
5. $y-\frac{1}{2}=\frac{1}{4}$	$y=\frac{3}{4}$.

Given Equations.	Answers.
6. $5+2x+3=3+x$	$x=-5$.
7. $x+4\pm10=0$	$x=-14$ or 6.
8. $7-3x+8+x=12-3x+8$	$x=5$.
9. $3x+22=31$	$x=3$.
10. $3x+6-x+3=16+x-4$	$x=3$.
11. $5v-9+4=0$	$v=1$.
12. $4+x-5-2x+2=5-2x+6$	$x=10$.
13. $\frac{x}{6}-1=0$	$x=6$.
14. $\frac{x}{2}-3=\pm1\frac{1}{2}$	$x=9$ or 3.
15. $x+\frac{3}{4}x=15$	$x=8\frac{4}{7}$.
16. $5x-15-2x-6=0$	$x=7$.
17. $4x+26=59-7x$	$x=3$.
18. $-\frac{4}{10}x-40+2x=0$	$x=25$.
19. $x+18=3x-5$	$11\frac{1}{2}=x$.
20. $2x-1=5x-19$	$6=x$.
21. $6x-\frac{1}{4}=\frac{1}{8}$	$x=\frac{7}{12}$.
22. $19x+13=59-4x$	$x=2$.
23. $z+15=-3z+35$	$z=5$.
24. $4x+12=2x+21-x$	$x=3$.
25. $2x-\frac{1}{2}x=6$	$x=4$.
26. $\frac{x}{6}-\frac{1}{6}=\frac{1}{7}$	$x=1\frac{6}{7}$.
27. $2y-\frac{4}{6}y-\frac{1}{2}-4=0$	$y=3\frac{6}{7}$.
28. $\frac{x+4}{3}-2=0$	$x=2$.
29. $\frac{2x}{3}+\frac{x}{4}=22$	$x=24$.
30. $\frac{x+4}{3}+\frac{x-2}{2}=7$	$x=8$.
31. $\frac{2x+8}{4-x}=6$	$x=2$.

Given Equations. ANSWERS.

32. $5(5x-6)-4(4x-5)+3(3x-2)-2x-16=0$ .. $x=2$.

33. $10(x+\frac{1}{2})-6x(\frac{1}{x}-\frac{1}{5})=23$ ..... $x=2$.

34. $\frac{x}{5}+\frac{x}{3}+7-x=0$ ..... $x=15$.

35. $\frac{1}{7}(x-\frac{1}{2})-\frac{1}{5}(\frac{3}{5}-x)=1\frac{13}{50}$ ..... $x=4\frac{7}{9}$.

36. $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}$ ..... $x=5$.

37. $-7=\frac{x}{4}-\frac{x}{2}-\frac{x}{3}$ ..... $x=12$.

38. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13$ ..... $x=12$.

39. $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}-44=0$ ..... $x=60$.

40. $\frac{x}{6}-8\frac{1}{5}=2(\frac{3x}{5}-1)-\frac{x+8}{3}+1\frac{2}{3}$ ..... $x=-8$.

41. $0=11-\frac{8x}{5}-4+\frac{9x}{10}$ ..... $x=10$.

42. $-1+\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{x}{6}=0$ ..... $x=12$.

43. $\frac{3x+30}{5}=66$ ..... $x=100$.

44. $\frac{x+1}{2}-\frac{x+4}{3}=1$ ..... $x=11$.

45. $\frac{2x+2}{3}-\frac{x+4}{6}+8=5$ ..... $x=-6$.

46. $\frac{x-5}{4}+6x=\frac{284-x}{5}$ ..... $x=9$.

47. $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$ ..... $x=7$.

48. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{5}$ ..... $x=19$.

49. $\frac{3x-\frac{2}{3}(1+x)}{4}+\frac{1-\frac{x}{5}}{5\frac{1}{2}}=\frac{2\frac{2}{3}+\frac{x-1}{2\frac{1}{2}}}{2\frac{1}{5}}$ ..... $x=2$.

Given Equations.	Answers.
50. $\frac{10+x}{5} : \frac{4x-9}{7} :: 14 : 5$	$x=4$.
51. $\frac{x}{2} - \frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \frac{5}{2} \cdot \frac{x^2+2}{3x-2}$	$x=4\frac{1}{5}$.
52. $\frac{1}{30-5x} = \frac{1}{3}$	$x=5$.
53. $\frac{x-1}{7} + \frac{3\frac{2}{7}-\frac{x}{7}}{\frac{5}{7}} = 7 - \frac{4+x}{4}$	$x=8$.
54. $8\frac{5}{4} + \frac{3x}{7} - \frac{x}{6} + 2x - 2\frac{2}{5}x + 13 + \frac{x}{4} = 0$	$x=-75\frac{10}{117}$.
55. $\frac{-5+x}{3} - \frac{2-x}{4} + 4 = \frac{25-x^2}{5+x}$	$x=2$.
56. $\frac{9x+20}{36} - \frac{x}{4} = \frac{4x-12}{5x-4}$	$x=8$.
57. $7-x-\frac{x+\frac{5}{2}}{\frac{9}{2}} = 6 - \frac{\frac{5x}{2}+\frac{1}{2}}{\frac{4}{3}}$	$x=3$.
58. $\frac{18x-19}{28} + \frac{11x+21}{6x+14} = \frac{9x+15}{14}$	$x=7$.
59. $\frac{9x-8}{7} + \frac{x+0.2}{0.4} + \frac{8}{9} = 0$	$x=\frac{1}{9}$.
60. $\frac{7.53x}{18} + 100 = \frac{2x}{5} + 3.86 - \frac{x}{6}$	$x=-519.675$ &c.

II. *Literal Equations.*

61. $bx=a$	$x=\frac{a}{b}$.
62. $\frac{y}{b} = 4d$	$y=4bd$.
63. $ay-b=by-c$	$y=\frac{b-c}{a-b}$.
64. $bx-6a=3c-2a-4x$	$x=\frac{4a+3c}{b+4}$.
65. $dx-5ac=3bc-x$	$x=\frac{5ac+3bc}{d+1}$.

Given Equations.	Answers.
66. $4b-3dy+7=4dy-ay+6+5b$	$y=\frac{b-1}{a-7d}$.
67. $\frac{1}{a}+\frac{b}{z}=c$	$z=\frac{ba}{ca-1}$.
68. $\frac{3a+x}{x}-5-\frac{6}{x}=0$	$x=\frac{3a-6}{4}$.
69. $6b+ax-1=6+7x+3b$	$x=\frac{7-3b}{a-7}$.
70. $\frac{4}{9}ab+\frac{4}{3}ac-\frac{2}{3}cx=\frac{5}{2}ac+2ab-6cx$ ..	$x=\frac{70ab-3ac}{320c}$.
71. $4a^2+4x^2=16a^2-16ax+4x^2$	$x=\frac{3}{4}a$.
72. $x+\frac{n}{m}x+\frac{p}{m}x=r$	$x=\frac{mr}{m+n+p}$.
73. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$	$x=\frac{3b}{3a-2b}$.
74. $\frac{a}{1+x}+\frac{a}{1-x}=b$	$x=\sqrt{\frac{b-2a}{b}}$.
75. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$	$x=\frac{abcd}{ab+ac+bc}$.
76. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$	$x=\frac{ac}{b}$.
77. $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$	$x=\frac{b}{c}$.
78. $6x+a : 4x+b=3x-b : 2x-a$	$x=\frac{b^2-a^2}{4a-b}$.
79. $\frac{ax}{b}+\frac{cx}{d}+\frac{ex}{f}-g=h$	$x=\frac{(g+h)bdf}{adf+bcf+bde}$.
80. $a-bx^{-2}=b+ax^{-2}$	$x=\sqrt{\frac{a+b}{a-b}}$.
81. $\frac{a}{b+x}+\frac{a}{b-x}=c$	$x=b\sqrt{\left(1-\frac{2a}{bc}\right)}$.
82. $x+1 : x-1 :: a+2b : a-3b$	$x=\frac{2a-b}{5b}$.

Given Equations. ANSWERS.

83. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}$. $x=\frac{ab}{a+b}$.

84. $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}-k=0$. $x=\frac{(ad+bc)fh+(eh+fg)bd}{bdfhk}$.

III. *Equations with surds.*

85. $\sqrt{x}=a$ ..... $x=a^2$.

86. $\sqrt{(x+6)}=3$ ..... $x=3$.

87. $\sqrt{x}-7=-3$ ..... $x=16$.

88. $\sqrt[3]{y}+5=11$ ..... $y=216$.

89. $(4+x)^{\frac{1}{2}}=5$ ..... $x=121$.

90. $(3y-5)^{\frac{1}{2}}=(7-y)^{\frac{1}{2}}$ ..... $y=3$.

91. $\sqrt{(x+a)}=\sqrt{x}+a$ ..... $x=\frac{1}{4}(a^2-2a+1)$.

92. $\sqrt[n]{(ax+b)}=\sqrt[n]{(cx+d)}$ ..... $x=\frac{d-b}{a-c}$.

93. $\sqrt{(x+5)}+3=8-\sqrt{x}$ ..... $x=4$.

94. $\sqrt{1+\sqrt{(3+\sqrt{6x})}}=2$ ..... $x=6$.

95. $\sqrt{(x+2)}\times\sqrt{5}=\sqrt{5x+2}$ ..... $x=\frac{9}{20}$.

96. $(12+x)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$ ..... $x=4$.

97. $\sqrt[4]{(11+\sqrt{5x})}-3=0$ ..... $x=980$.

98. $\sqrt{(2+x)}+\sqrt{x}=\frac{4}{\sqrt{(2+x)}}$ ..... $x=\frac{2}{3}$.

99. $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{9}\sqrt{a}$ ..... $x=\frac{25}{16}a$.

100. $\sqrt[4]{\{-1+\sqrt[3]{(\sqrt{7x}+2)}\}}=1$ ..... $x=5\frac{1}{7}$.

101. $\sqrt{\frac{a^2-x^2}{a-x}}+x-a=2x$ ..... $x=1-a$.

102. $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$ ..... $x=\frac{1}{1-a}$.

Given Equations.	Answers.
103. $\sqrt{(a+x)}+\sqrt{(a-x)}=b$	$x=\frac{1}{2}b\sqrt{(4a-b^2)}$.
104. $(7+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=\frac{8}{(7+x)^{\frac{1}{2}}}$	$x=\frac{1}{9}$.
105. $\sqrt{[11+\sqrt[3]{-13-\sqrt{20+\sqrt[4]{(3x+25)}}}]}=3$.	$x=200$.
106. $(x+x^{\frac{1}{2}})^{\frac{1}{2}}-(x-x^{\frac{1}{2}})^{\frac{1}{2}}=\frac{2}{3}\left(\frac{x}{x+x^{\frac{1}{2}}}\right)^{\frac{1}{2}}$	$x=\frac{25}{16}$.
107. $\frac{3x-16}{\sqrt{3x+4}}=a+\frac{\sqrt{3x-4}}{a}$	$x=\frac{1}{3}\left(4+\frac{a^2}{a-1}\right)^2$.
108. $\sqrt{(a+x)}+\sqrt{\frac{a}{x}}=\sqrt{x}$	$x=\frac{a}{a+2\sqrt{a}}$.
109. $(x^2-b^2)^{\frac{1}{2}}=(a-x)^{\frac{1}{2}}$	$x=\frac{a^2+b^2}{2a}$.
110. $(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}=3(a^2-x^2)^{\frac{1}{2}}$	$x=\frac{2}{5}a\sqrt{5}$
111. $\frac{3x-1}{\sqrt{3x+1}}=1+\frac{\sqrt{3x-1}}{2}$	$x=3$.
112. $\frac{\sqrt{ax}-b}{\sqrt{ax}+3b}=\frac{\sqrt{ax}-2b}{\sqrt{ax}+5b}$	$x=\frac{1}{9}\cdot\frac{b^2}{a}$.
113. $\frac{x^{\frac{1}{2}}+b^{\frac{1}{2}}}{x^{\frac{1}{2}}-b^{\frac{1}{2}}}=\frac{a}{b}$	$x=b\left(\frac{a+b}{a-b}\right)^2$.
114. $(a+x)^{\frac{1}{n}}=(x^2+5ax+b^2)^{\frac{1}{2n}}$	$x=\frac{a^2-b^2}{3a}$
115. $\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}=\sqrt[3]{2}$	$x=1$.
116. $\frac{\sqrt[3]{(x+1)}-\sqrt[3]{(x-1)}}{\sqrt[3]{(x+1)}+\sqrt[3]{(x-1)}}=\frac{1}{2}$	$x=1\frac{1}{13}$.
117. $\frac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b^2$	$x=\frac{a}{2b}(b-1)^2$.
118. $\sqrt{(1-x)}+\sqrt{(1-x+\sqrt{1+x})}=\sqrt{(1+x)}$ ...	$x=\frac{24}{25}$.
119. $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a$.	$x=8$.
120. $(a^2-x^2)^{\frac{1}{2}}+x(a^2-1)^{\frac{1}{2}}=a^2(1-x^2)^{\frac{1}{2}}$ ..	$x=\left(\frac{a^2-1}{a^2+3}\right)^{\frac{1}{2}}$.

Given Equations.	Answers.
121. $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$	$x=a\left(\frac{b+c}{b-c}\right)$.
122. $\left[\frac{x+1}{x-1}\right]^{\frac{1}{2}}+\left[\frac{x-1}{x+1}\right]^{\frac{1}{2}}=a$	$x=\frac{a}{(a^2-4)^{\frac{1}{2}}}$.
123. $\sqrt{(4a+x)}=2\sqrt{(b+x)}-\sqrt{x}$	$x=\frac{(b-a)^2}{2a-b}$.*

SIMPLE EQUATIONS WITH MORE THAN ONE UNKNOWN QUANTITY.

128. When, in the expression for one unknown quantity, another unknown quantity occurs, it is plain (Art 122) that the value of the former cannot be determined unless that of the latter be known. Thus, the equation

$$ax+by=c$$

being solved with respect to x, gives

$$x=\frac{c-by}{a}:$$

where, since a, b, c, denote known quantities, x will de-

* Numerical equations of such general forms as Ex. 72, 75, 79, 84, may readily be solved by substituting particular numbers for the letters, in the value of x: thus, if the equation

$$\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6}-366=0, \text{ be given,}$$

we have by substitution in Ex. 79,

$$x=\frac{(366+0)\times 360}{270+168+660}=\frac{366\times 360}{1098}=\frac{366\times 20\times 18}{61\times 18}=\frac{366\times 20}{61}=6\times 20=120.$$

Also, $\frac{12}{5x}+\frac{9}{4x}+\frac{20}{7x}+\frac{3}{2x}-87=0$ gives, by Ex. 84.

$$x=\frac{1302+1221}{280\times 87}=\frac{2523}{24360}=\frac{29}{280}.$$

pend for its value upon y; so that, whatever value is assigned to y, x will assume a corresponding value.

Again, if the equation be solved with respect to y, it will give

$$y = \frac{c - ax}{b};$$

and y being expressed in terms of x, it will be dependent upon it, and may have an unlimited number of values, corresponding to those assigned to x. It is plain, therefore, that other conditions must be given, in order to enable us to determine the value of either unknown, and consequently that of the other: such conditions are given by other equations, which hold true simultaneously with the given one, and are hence called *simultaneous* equations.

To illustrate this, let us take the equation

$$x + y = 12.$$

Here, the only condition laid down is, that the sum of the two unknowns must be 12; and it is obvious that the equation can be satisfied by a great many numbers;

If $x = 5$,	$y = 7$	If $x = 16$,	$y = -4$
$x = 2$,	$y = 10$	$x = -3$,	$y = 15$
$x = 1$,	$y = 11$	$x = -1\frac{1}{2}$,	$y = 13\frac{1}{2}$
$x = 3\frac{1}{2}$,	$y = 8\frac{1}{2}$	$x = 8$,	$y = 4$
$x = 2\frac{1}{4}$,	$y = 9\frac{3}{4}$	$x = 14\frac{3}{7}$	$y = -2\frac{3}{7}$
&c.		&c.	

Let us now introduce another condition, namely, that the difference of the same unknown quantities must be 4, or that

$$x - y = 4:$$

then x and y become at once determinate, and are respectively 8 and 4. If the equation $x - y = 4$ be considered

by itself there is an infinite variety of numbers which would satisfy it. Thus,

If $x = 20$,	$y = 16$	If $x = -3$,	$y = -7$
$x = 7$,	$y = 3$	$x = -1\frac{1}{2}$,	$y = -5\frac{1}{2}$
$x = \frac{1}{2}$,	$y = -3\frac{1}{2}$	$x = 8$	$y = 4$
&c.		&c.	

Among all the above solutions, there is only one, namely $x = 8$, $y = 4$, which satisfies both equations at once.

129. When each equation contains a separate condition, not fulfilled by another equation, nor derivable from it, the equations are called *independent*. Thus the equations

$$x + y = 12,\ x - y = 4,$$

are independent of one another, as they involve separate conditions: the equation $2x + 2y = 24$ does not involve any new condition, separate from that given by $x + y = 12$, and is not therefore independent of it.

In like manner, in the following equations,

$$\begin{aligned} x + 2y - 3z &= 15; \\ 3x - 4y + z &= 5; \\ 4x - 2y - 2z &= 20; \end{aligned}$$

the first and second involve separate conditions, but the third is the sum of the other two; it contains therefore no new condition: it is not independent, and consequently is of no use in determining the values of the unknown quantities.

From this we may conclude, that in order to determine the values of unknown quantities, *there must be as many conditions as there are unknown quantities, and therefore as many equations as there are unknown quantities.**

* If the independent equations be fewer in number than the unknown quantities, there are too few conditions; and if they are

130. PROB. I. *To solve simple equations containing two unknown quantities.* (1) If the coefficients of one of the unknown quantities be not the same in both equations, multiply or divide one or both equations by such numbers as will render those coefficients the same; (2) destroy the terms thus rendered identical, by adding or subtracting the equations, and one of the unknown quantities will be eliminated; (3) from the resulting equation, find the value of the unknown quantity which it contains; (4) then the value of the unknown, which was eliminated, will be determined, either by repeating a similar process upon the given equations, or by substituting, in either of them, the value of the unknown already found.

1. Given, $2x+y=35$
$5x-3y=27.$

It is here obvious, that if the first equation be multiplied by 3, y may be at once eliminated. Multiplying, we find

$6x+3y=105;$
$5x-3y=27;$

adding we have $11x=132;$

therefore, $x=12.$

But, from the first equation, $y=35-2x;$

more numerous, they may give conditions which are impossible, or incongruous, that is inconsistent with one another; and, when differently combined, may give different values of the unknown; or, though they should not happen to give such conditions, they are not required for the solution, and are therefore useless.

In a series of connected equations, it is not necessary that each equation should contain all the unknown quantities which are to be determined: but merely that each equation of the series should give a separate condition.

whence, $y=35-24=11$;

substitution being in this case the more simple method.

2. Given, $5x+7y=201$,

$8x-3y=137$.

Here, to eliminate y, multiply each equation by the coefficient of y in the other, and there will be obtained

$15x+21y=603$,

$56x-21y=959$;

adding, $71x=1562$;

whence, $x=22$.

But, from the second equation,

$$y=\frac{8x-137}{3},$$

or $$y=\frac{176-137}{3}=\frac{39}{3}=13.$$

3. Given the equations

$$\frac{2x}{3}-4+\tfrac{1}{12}y+x=8-\tfrac{5}{4}y+1\tfrac{1}{2},$$

$$\tfrac{1}{6}y+\tfrac{1}{2}x+2=\tfrac{1}{6}-2x+6.$$

Here, 12 and 6 are the least common multiples of the denominators; therefore

$$8x-48+6y+12x=96-9y+1,$$

$$y-3x+12=1-12x+36.$$

By reduction, these become

$$20x+15y=145,$$

$$9x+y=25.$$

Dividing the terms of the first by 5, and multiplying those of the second by 3, we obtain

$$4x+3y=29,$$

$$27x+3y=75.$$

Taking the first from the second, the remainder,

$$23x = 46,$$

gives $x = 2$;

but $y = 25 - 9x$,

therefore $y = 7$.

RULE II.

131. (1) Find an expression for either of the unknown quantities from both the equations; (2) and since the unknowns have the same value in each, equate the two expressions thus found, and from the resulting equation find the value of the unknown quantity which it contains; (3) substitute this value in either of the expressions before found for the other unknown.

EXAMPLE.

1. Given, $2x + y = 35$

$5x - 3y = 27$

to find the values of x and y.

The solution will be more easily effected by finding y from both equations: this gives

$$y = 35 - 2x,$$

and $$y = \frac{5x - 27}{3};$$

therefore $$35 - 2x = \frac{5x - 27}{3};$$

whence we readily find

$$x = 12.$$

Substituting this value of x in the first expression for y, which is the simpler, we find

$$y = 35 - 24 = 11.$$

If we first find x from both equations, we shall have

$$\frac{35-y}{2} = \frac{27+3y}{5};$$

whence $$y = 11.$$

Then, by substitution in the first expression for x,

$$x = \frac{35-11}{2} = 12.$$

RULE III.

132. (1) Find an expression for either of the unknown quantities, in one of the equations; (2) substitute it instead of that unknown in the other equation; (3) from the resulting equation, find the value of the unknown quantity which it contains; (4) and the value of the other will be determined by substitution.

EXAMPLES.

1. Given, $$2x+y = 35,$$
$$5x-3y = 27.$$

From the first,

$$x = \frac{35-y}{2};$$

putting this in the second, we have

$$5 \times \frac{35-y}{2} - 3y = 27.$$

by multiplication, $175-5y-6y = 54$;

whence $$121 = 11y,$$

and $$11 = y;$$

consequently $$x = \frac{35-11}{2} = 12.$$

2. Given, $$x+y = a,$$
$$x^2-y^2 = b.$$

Here, from the first equation,

$$x = a - y,$$

which being put in the second, gives

$$(a-y)^2 - y^2 = b,$$

or $$a^2 - 2ay + y^2 - y^2 = b;$$

whence $$a^2 - 2ay = b,$$

and $$y = \frac{a^2 - b}{2a};$$

then, by substitution,

$$x = a - \frac{a^2 - b}{2a};$$

by reduction,

$$x = \frac{a^2 + b}{2a}.$$

RULE IV.

133. (1) Multiply one of the equations by any assumed quantity; (2) from the product, subtract the other equation; (3) and, in the result, assign such values to the assumed quantity as will make first one, and then the other, unknown quantity vanish.

Given, $$2x + 3y = 23,$$

$$5x - 2y = 10.$$

Multiplying the terms of the first by an assumed quantity, m, we have

$$2mx + 3my = 23m;$$

from this take

$$5x - 2y = 10,$$

and there remains

$$(2m-5)x + (3m+2)y = 23m - 10.$$

Now, m being arbitrary, we may take it such that x shall vanish: we may therefore put $2m-5=0$; whence, $2m=5$, and $m=2\frac{1}{2}$: we shall therefore have

$$(5-5)x+(3\times 2\tfrac{1}{2}+2)y=23\times 2\tfrac{1}{2}-10;$$

or simplifying $9\frac{1}{2}y=47\frac{1}{2}$,

that is, $19y=95$;

whence $y=5$.

Again, to make y disappear from the same equation, we must have $3m+2=0$; therefore $m=-\frac{2}{3}$: hence,

$$(-\tfrac{4}{3}-5)x=-\tfrac{46}{3}-10;$$

by addition, $19x=76$;

whence $x=4$.

134. With respect to the comparative advantages of these different methods, it may be observed, that the first is in general the most convenient: the second and third almost always give rise to fractions, from which the reduced equation must be afterwards cleared. In the first method, also, the addition and subtraction can be performed, when the coefficients are not large numbers, at the same time with the multiplication or division, which shortens the process considerably. The fourth method will often be found the most expeditious.

EXERCISES.

Given Equations.		Answers.
1.	$x+y=5$,	$x=3$.
	$x-y=1$.	$y=2$.
2.	$x+y=19$,	$x=7$.
	$y-x=5$.	$y=12$.
3.	$3x-2y=14$,	$x=8$.
	$x+y=13$.	$y=5$.

Given Equations.		Answers.
4.	$4x+2y=26,$	$x=4.$
	$2x-y=3.$	$y=5.$
5.	$3y+6x=15,$	$x=2.$
	$6x-3y=9.$	$y=1.$
6.	$5y-5x=15,$	$x=7.$
	$3x+5y=71.$	$y=10.$
7.	$-4x+3y=13,$	$x=5.$
	$2y-6x=-8.$	$y=11.$
8.	$5x-2y+4=0,$	$x=4.$
	$4x+y-28=0.$	$y=12.$
9.	$10x=2+2y,$	$x=1.$
	$4y=20-4x.$	$y=4.$
10.	$ax+by=c,$	$x=\frac{cf-bh}{af-bd}.$
	$dx+fy=h.$	$y=\frac{ah-dc}{af-bd}.$
11.	$x+y=s$	$x=\frac{1}{2}(s+d).$
	$x-y=d$	$y=\frac{1}{2}(s-d).$
12.	$3x+2y=4$	$x=1.$
	$4y-3x+1=0$	$y=\frac{1}{2}.$
13.	$ax=by,$	$x=\frac{bc}{a+b}.$
	$x+y=c$	$y=\frac{ac}{a+b}.$
14.	$5v-2z=\frac{1}{3},$	$v=\frac{2}{3}.$
	$6z-6v=5.$	$z=\frac{5}{2}.$
15.	$x+ny=p$	$x=\frac{nq-p}{n^2-1}.$
	$y+nx=q$	$y=\frac{np-q}{n^2-1}.$

Given Equations.	Answers.
16. $\frac{x}{2}+\frac{y}{4}=7,$	$x=10.$
$\frac{x}{3}+\frac{y}{3}=6.$	$y=8.$
17. $3x-7=4+x+y,$	$y=103.$
$2y+79=5x.$	$x=57.$
18. $px+y=1,$	$x=\frac{p+2}{1+p^2}.$
$x-py=2.$	$y=\frac{1-2p}{1+p^2}.$
19. $2x+.4y=1.2$	$x=.02.$
$3.4x-.02y=.01$	$y=2.9.$
20. $ax+by=n,$	$x=\frac{4n-m}{6a}.$
$4by-2ax=m.$	$y=\frac{2n+m}{6b}.$
21. $x+1=\sqrt{(x^2+2y-1)}$	$x=3.$
$y+1=\sqrt{(3x+y^2)}$	$y=4.$
22. $\frac{100-x}{3}=\frac{100+y}{4}$	$x=10.$
$\frac{50-x}{4}=\frac{50+y}{7}$	$y=20.$
23. $\frac{x}{8}+8y=194,$	$x=16.$
$\frac{y}{8}+8x=131.$	$y=24.$
24. $(x+5)(y+7)=(x+1)(y-9)+112,$	$x=3.$
$2x+10=3y+1.$	$y=5.$
25. $\frac{m}{x}+\frac{n}{y}=a$	$x=\frac{m^2-n^2}{ma-nb}.$
$\frac{n}{x}+\frac{m}{y}=b$	$y=\frac{m^2-n^2}{mb-na}.$

Given Equations.	Answers.
26. $\frac{a}{x}+\frac{b}{y}=m,$ $\frac{c}{x}+\frac{d}{y}=n.$	$x=\frac{bc-ad}{nb-md}.$ $y=\frac{bc-ad}{mc-na}.$
27. $x+y+\frac{3}{10}=1,$ $2-5y+10x=0.$	$x=\frac{1}{10}.$ $y=\frac{3}{5}.$
28. $\frac{4x-8y+5}{2}=\frac{10x^2-12y^2-14xy+2x}{5x+3y+3}+2,$ $\sqrt{(6+x)}:\sqrt{(6-y)}::3:2$	$x=3.$ $y=2.$
29. $2x+3y=\frac{4a}{b},$ $y-x=\frac{a}{12b}.$	$x=\frac{3}{4}\frac{a}{b}.$ $y=\frac{5}{6}\frac{a}{b}.$
30. $\frac{1}{x}+\frac{1}{y}-7=3,$ $\frac{2}{x}-\frac{2}{y}+7=-5.$	$x=\frac{1}{2}.$ $y=\frac{1}{8}.$
31. $x+\sqrt{(x^2+y)}=a,$ $x+\sqrt{(x^2-y)}=b.$	$x=\frac{a^2+b^2}{2(a+b)}.$ $y=\frac{ab(a-b)}{a+b}.$
32. $\frac{3x-1}{5}+3y-4=15,$ $\frac{3y-5}{6}+2x-8=7\frac{2}{3}.$	$y=5.$ $x=7.$
33. $\frac{a}{b+y}=\frac{b}{3a+x},$ $ax+2by=c.$	$x=\frac{2b^2-6a^2+c}{3a}.$ $y=\frac{3a^2-b^2+c}{3b}.$

Given Equations.	Answers.
34. $5x+3y=\frac{(8a-2b)ab}{a^2-b^2}$,	$x=\frac{ab}{a-b}$.
$a^2y-\frac{acb^2}{a+b}+(a+b+c)bx=b^2y+(a+2b)ab$.	$y=\frac{ab}{a+b}$.
35. $\frac{3x-5y}{3}-\frac{2x-8y-9}{12}=\frac{1}{2}y+\frac{1}{3}+\frac{1}{4}$,	$x=7$.
$\frac{x}{7}+\frac{y}{4}+1\frac{1}{5}:4x-\frac{y}{8}-24::3\frac{1}{3}:3\frac{1}{2}$.	$y=4$.
36. $x+y=18.73$,	$x=-38.084$.
$0.56x+13.8121y=763.4$.	$y=56.814$.
37. $x+y:a::x-y:b$,	$x=\frac{1}{2}(a+b)\sqrt{\frac{c}{ab}}$.
$x^2-y^2=c$,	$y=\frac{1}{2}(a-b)\sqrt{\frac{c}{ab}}$.
38. $bcx=cy-2b$,	$x=\frac{a}{bc}$.
$b^2y+\frac{a(c^3-b^3)}{bc}=\frac{2b^3}{c}+c^2x$.	$y=\frac{a+2b}{c}$.
39. $4x-34\frac{1}{3}-\frac{4y+13x}{27-6y}=\frac{12x+8}{3}$,	$x=7$.
$3x+\frac{21-4y}{4x-10}=\frac{18x+13}{6}-2\frac{1}{9}$.	$y=5$.
40. $\frac{7x+6}{11}+\frac{4y-9}{3}=3x-\frac{13-x}{2}-\frac{3y-x}{5}$,	$x=7$.
$3x+4:2y-3::5:3$.	$y=9$.
41. $\frac{\frac{7x}{4}+6y}{5}-\frac{\frac{3y+6}{5}-\frac{3x-2}{10}}{8}=5-\frac{x}{16}$	$x=4$.
$\frac{3x}{2}+\frac{2y}{3}+2\frac{1}{2}:\frac{x}{2}-\frac{y}{3}+\frac{1}{6}::10\frac{1}{2}:1\frac{1}{6}$.	$y=3$.

Given Equations. Answers.

42. $4+\dfrac{12y-\dfrac{6y+2}{x}}{11}=y+\dfrac{\dfrac{3xy-31}{11}+10x+13}{3x}$, $x=7$.

$\dfrac{2x}{3}-\dfrac{3x-5}{y+7}=\dfrac{4xy+\frac{170}{3}}{6y+27}$. $y=2$.

43. $\dfrac{4x-2y+3}{3}-\dfrac{18-x+5y}{7}=\dfrac{x}{4}-\dfrac{y}{5}-\frac{1}{7}-7\frac{7}{15}$.

$2x-y+15:y-2x+15::\dfrac{x}{3}-\dfrac{y}{4}+\frac{5}{4}:\dfrac{y}{4}-\dfrac{x}{3}+\frac{1}{12}$.

$x=18$.

$y=24$.

44. $\dfrac{x}{a}+\dfrac{y}{b}=1-\dfrac{x}{c}$, $x=\dfrac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

$\dfrac{y}{a}+\dfrac{x}{b}=1+\dfrac{y}{c}$ $y=\dfrac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

45. $\sqrt{y}-\sqrt{(20-x)}=\sqrt{(y-x)}$

$\sqrt{(y-x)}+\sqrt{(20-x)}:\sqrt{(y-x)}=5:3$, $x=16$.

$y=25$.

135. Prob. II. *To solve simple equations containing three unknown quantities.*

Rule I.

(1) Eliminate, by the rules of the last Problem, one unknown quantity from two of the equations: (2) and then the same unknown from the third equation, and either of the other two; (3) and from the two resulting equations, find the values of the two unknown quantities which they contain.

EXAMPLES.

1. Given, $5x-6y+4z=15$, (1)
$7x+4y-3z=19$, (2).
$2x+y+6z=46$. (3)

Here z will be most easily eliminated: to effect this, multiply equation (1) by 3, (2) by 4, add the results, and there is obtained

$$43x-2y=121. \quad (4)$$

Again, doubling (2), and adding the result to (3), we find

$$16x+9y=84. \quad (5)$$

Solving (4) and (5) by any of the Rules of the last Problem, we find

$$x=3,\ y=4;$$

and hence, by substitution in (3), $z=6$.

2 Given, $x+2y-3z=15$, (1)
$3x-4y+z=5$, (2)
$x+3y+3z=40$. (3)

From equation (1),

$$x=15-2y+3z;$$

which being substituted (Art. 132) in (2) and (3), gives

$$45-6y+9z-4y+z=5,$$
$$15-2y+3z+3y+3z=40.$$

By reduction, these become

$$10y-10z=40,$$

or $$y-z=4,$$

and $$y+6z=25;$$

whence we find (Art. 130)

$$z=3,\ y=7,$$

and therefore

$$x=10.$$

3. Given,
$$x+y+z=29,$$
$$x+2y+3z=62,$$
$$\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z=10.$$

From these we find
$$x=29-y-z, \quad (1)$$
$$x=62-2y-3z, \quad (2)$$
$$x=20-\tfrac{2}{3}y-\tfrac{1}{2}z \quad (3)$$

Equating (1) and (2), and also (1) and (3), and reducing, we find
$$y+2z=33,$$
$$2y+3z=54;$$

from which we deduce
$$z=12,\ y=9.$$

Hence
$$x=8.$$

RULE II.

136. (1) Multiply two of the equations by any two assumed quantities; (2) from these products, subtract the remaining equation; (3) in the result, assign successively such values to the assumed quantities as will make two of the unknowns vanish at once; (4) in the expressions for the unknown quantities deduced from the three equations thus found, substitute the values of the assumed quantities, determined from the condition that the coefficients must vanish; and the values of the unknowns will be exhibited in known terms.

EXAMPLES.

1. Given,
$$ax+by+cz=d, \quad (1)$$
$$a'x+b'y+c'z=d', \quad (2)$$
$$a''x+b''y+c''z=d''. \quad (3)$$

Multiply the first by any assumed quantity, m; the second by any other quantity, n: then adding the results, and subtracting the third equation, we find

$$(am + a'n - a'')x + (bm + b'n - b'')y + (cm + c'n - c'')z = dm + d'n - d''. \quad (4)$$

Since m and n are arbitrary. they may be taken of such value that x and y shall vanish from this equation. Now, in order that this may take place, their coefficients must become zero; that is, we must have

$$am + a'n = a'', \quad (5)$$
$$bm + b'n = b''. \quad (6)$$

On this hypothesis, equation (4) will become

$$(cm + c'n - c'')z = dm + d'n - d'';$$

whence,
$$z = \frac{dm + d'n - d''}{cm + c'n - c''}.$$

From equations (5) and (6) m and n can be readily determined by any of the Rules of Prob. I.: these values are

$$m = \frac{a''b' - a'b''}{ab' - a'b},$$
$$n = \frac{ab'' - a''b}{ab' - a'b}.$$

By substituting these values for m and n in that of z, and then reducing, the latter becomes

$$z = \frac{d(b'a'' - a'b'') + d'(ab'' - ba'') - d''(ab' - ba')}{c(b'a'' - a'b'') + c'(ab'' - ba'') - c''(ab' - ba')}.$$

Again, to make x and z vanish, we must have

$$am + a'n = a'',$$
$$cm + c'n = c''.$$

On this supposition we find, from equation (4),

$$(bm + b'n - b'')y = dm + d'n - d'';$$

whence, $$y = \frac{dm + d'n - d''}{bm + b'n - b''};$$

the values of m and n, in this case, are

$$m = \frac{a''c' - a'c''}{ac' - a'c},$$

$$n = \frac{ac'' - a''c}{ac' - a'c}.$$

Substituting these as before in the value of y, and reducing the compound fraction, that value becomes

$$y = \frac{d(c'a'' - a'c'') + d'(ac'' - ca'') - d''(ac' - ca')}{b(c'a'' - a'c'') + b'(ac'' - ca'') - b''(ac' - ca')}.$$

Lastly, to make y and z disappear, we require the conditions,

$$bm + b'n = b'',$$
$$cm + c'n = c''.$$

Proceeding as before, we shall find

$$x = \frac{dm + d'n - d''}{am + a'n - a''};$$

and by substitution of the values which m and n have in this case

$$x = \frac{d(c'b'' - b'c'') + d'(bc'' - cb'') - d''(bc' - cb')}{a(c'b'' - b'c'') + a'(bc'' - cb'') - a''(bc' - cb')}.$$

137. These three formulæ may be put under more symmetrical forms, by performing the actual multiplications, arranging the terms with respect to the accents upon a, and so as to make the terms alternately positive and negative, and by changing all the signs in the numerator and denominator of the values of x and z, which will not alter those values: the forms thus deduced are

$$x = \frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

EXAMPLE.

Given,
$$7x+5y+2z = 79,$$
$$8x+7y+9z = 122,$$
$$x+4y+5z = 55,$$

to find x, y, z.

Comparing these with the three general equations at the beginning of the last Art. we perceive that in this case

$$a = 7, \quad b = 5, \quad c = 2, \quad d = 79;$$
$$a' = 8, \quad b' = 7, \quad c' = 9, \quad d' = 122;$$
$$a'' = 1, \quad b'' = 4, \quad c'' = 5, \quad d'' = 55.$$

Substituting these numbers in the general formulæ, and actually multiplying, we have

$$x = \frac{2765 - 2844 + 976 - 3050 + 2475 - 770}{245 - 252 + 64 - 200 + 45 - 14} = \frac{-448}{-112} = 4.$$

The values of the two other unknowns, having the same denominator as that of x, are found more readily: they are

$$y = 9, \quad z = 3.$$

When one unknown is thus determined, the others may be found by the Rules of Prob. I Art. 130.

138. Either of the Rules of the last Problem may be applied, with slight modifications, to equations containing four or more unknown quantities. Thus, by Rule II. each of the given equations, except the last, may be multiplied by an assumed quantity, as m, n, p, &c.; and from the sum, the last being subtracted, m, n, p, &c. may be determined so as to make all the unknowns but one

vanish. The value of each may be thus found; or, according to Rule I. the unknowns may be eliminated one by one. This latter method is generally to be preferred to any other, in solving particular examples.

EXAMPLES.

1. Given,
$$2x-3y+2z=13, \quad (1)$$
$$4u-2x=30, \quad (2)$$
$$4y+2z=14, \quad (3)$$
$$5y+3u=32. \quad (4)$$

Here it is evident that, if the difference of (1) and (3) be taken, z will be eliminated: this will give

$$7y-2x=1; \quad (4)$$

and that, if equation (2) be multiplied by 3, and equation (4) by 4, and the difference of the products taken, u will be eliminated: we thus get

$$20y+6x=38. \quad (5)$$

Combining this equation with (4), we find

$$y=1,$$

and
$$x=3,$$

also
$$4u-2x=30;$$

hence we find, by substitution,

$$u=9, \text{ and } z=5.$$

2. Given,
$$7x-2z+3u=17, \quad (1)$$
$$4y-2z+t=11, \quad (2)$$
$$5y-3x-2u=8, \quad (3)$$
$$4y-3u+2t=9, \quad (4)$$
$$3z+8u=33, \quad (5)$$

Here, eliminating z from equations (1) and (5), we find

$$21x+25u=117;$$

whence, by division,

$$x = \frac{117 - 25u}{21}.$$

Substituting this value of x in equation (3), we obtain

$$5y - \frac{117 - 25u}{7} - 2u = 8;$$

which becomes, by reduction,

$$35y + 11u = 173. \quad (6)$$

Again, in order to find another equation which may be combined with this one, so as to determine either u or y, substitute in equation (2) the value of z, deduced from equation (5); and, after reduction, there will be obtained

$$12y + 16u + 3t = 99.$$

Combining this with equation (4), and eliminating t, there is obtained

$$12y + 41u = 171. \quad (7)$$

From this last equation and (6), we readily find the value of u to be

$$u = 3;$$

whence, from (6),

$$y = 4;$$

also, from equation (5),

$$z = \tfrac{1}{3}(33 - 24) = 3;$$

from equation (2) or (4),

$$t = 1;$$

and from (1) or (3),

$$x = 2.$$

EXERCISES.

Given Equations.	Answers
1. $x + y - z = 23,$	$x = 20.$
$x - y + z = 17,$	$y = 15.$
$x + y + z = 47.$	$z = 12.$

Given Equations.	Answers.
2. $x+y+z=33$,	$x=5$.
$y-x+z=23$,	$y=11$.
$z-x-y=1$.	$z=17$.
3. $x+y+z=31$,	$x=20$.
$x+y-z=25$,	$y=8$.
$x-y-z=9$.	$z=3$.
4. $x+y+z=29\frac{1}{4}$,	$x=16$.
$x+y-z=18\frac{1}{4}$,	$y=7\frac{3}{4}$.
$x-y+z=13\frac{3}{4}$,	$z=5\frac{1}{2}$.
5. $x+y+z=53$,	$x=23$.
$x+2y+3z=107$,	$y=6$.
$x+3y+4z=137$,	$z=24$.
6. $x+2y+z=16$,	$x=3$.
$\frac{x}{3}+\frac{y}{2}+\frac{z}{5}=4$,	$y=4$.
$2x+4y-3z=7$.	$z=5$.
7. $v+4u+3w=18$,	$v=1$.
$3v-2u+4w=11$,	$u=2$.
$4v+6u-5w=1$.	$w=3$.
8. $2v+4u-3w=22$,	$v=3$.
$4v-2u+5w=18$,	$u=7$.
$6v+7u-w=63$.	$w=4$.
9. $x+y=10$,	$x=3$.
$x+z=19$,	$y=7$.
$y+z=23$.	$z=16$.
10. $x-z=1$,	$x=10$.
$2y+z=11$,	$y=1$.
$2x+3y=23$.	$z=9$.

Given Equations.	Answers.
11. $x+y+z=30$,	$x=\frac{2}{3}$.
$8x+4y+2z=50$,	$y=-7$.
$27x+9y+3z=64$.	$z=36\frac{1}{3}$.
12. $4x+12y+7x=128$,	$z=3$.
$7z+3y+3x=60$,	$y=5$.
$5z+y+6x=68$.	$x=8$.
13. $4z-5y=35$,	$y=17$.
$7x-2y=22$,	$x=8$.
$2z+x=68$.	$z=30$.
14. $3x-41-9y+8z=0$,	$x=2$.
$-5x+4y+20+2z=0$,	$y=-3$.
$11x-7y-37-6z=0$.	$z=1$.
15. $3z=2x$,	$x=12$.
$2x+y=34$,	$y=10$.
$y+z=18$.	$z=8$.
16. $\frac{x+y}{3}+2z=21$,	$x=24$.
$\frac{y+z}{2}-3x=-65$,	$y=9$.
$\frac{3x+y-z}{2}=38$,	$z=5$.
17. $\frac{1}{x}+\frac{1}{y}=a$,	$x=\frac{2}{a+b-c}$.
$\frac{1}{x}+\frac{1}{z}=b$,	$y=\frac{2}{a+c-b}$.
$\frac{1}{y}+\frac{1}{z}=c$.	$z=\frac{2}{b+c-a}$.

Given Equations.	Answers.
18. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4}$,	$x=2.$
$\frac{1}{x}+\frac{1}{z}=\frac{2}{3}$,	$y=4.$
$\frac{1}{y}+\frac{1}{z}=\frac{5}{12}$.	$z=6.$*
19. $x+\frac{1}{2}y+\frac{1}{3}z=32$,	$x=12.$
$\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$,	$y=20.$
$\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12.$	$z=30.$
20. $my=nx$,	$x=\frac{amp}{np+m(p+q)}.$
$pz=qx$,	$y=\frac{anp}{np+m(p+q)}.$
$x+y+z=a.$	$z=\frac{amq}{np+m(p+q)}.$
21. $ax+by=c$,	$x=\frac{ce-bf}{ae-bd}.$
$dx+ey=f$,	$y=\frac{af-cd}{ae-bd}.$
$gy+hz=l.$	$z=\frac{a(el-fg)-d(bl-cg)}{h(ae-bd)}.$
22. $x+y=a$,	$x=\frac{1}{2}(a+b-c).$
$x+z=b$,	$y=\frac{1}{2}(a+c-b).$
$y+z=c.$†	$z=\frac{1}{2}(b+c-a).$
23. $x-9y+3z-10u=21$,	$x=100.$
$2x+7y-z-u=683$,	$y=60.$
$3x+y+5z+2u=195$,	$z=-13.$
$4x-6y-2z-9u=516.$	$u=-50.$

* This equation may also be solved by substituting for a, b, &c. in the last equation, their numerical values in this one.

† This equation is most easily solved, by subtracting each equation severally from the half sum of the three,

Given Equations.	Answers.
24. $x-y=2$,	$x=7$.
$x-z=3$,	$y=5$.
$y+z=9$.	$z=4$.
25. $\frac{xy}{x+y}=1$,	$x=\frac{12}{7}$.
$\frac{xz}{x+z}=2$,	$y=\frac{12}{5}$.
$\frac{yz}{y+z}=3$.	$z=-12$.
26. $5x-11\sqrt{y}+13\sqrt[3]{z}=22$,	$x=1$.
$4x+6\sqrt{y}+5\sqrt[3]{z}=31$,	$y=4$.
$x-\sqrt{y}+\sqrt[3]{z}=2$.	$z=27$.
27. $9x-2z+u=41$,	$x=5$.
$7y-5z-t=12$,	$y=4$.
$4y-3x+2u=5$,	$z=3$.
$3y-4u+3t=7$,	$u=2$.
$7z-5u=11$.	$t=1$.
28. $xy=a(x+y)$	$x=\frac{2abc}{ac+bc-ab}$.
$xz=b(x+z)$	$y=\frac{2abc}{ab+bc-ac}$.
$yz=c(y+z)$	$z=\frac{2abc}{ab+ac-bc}$.
29. $x(x+y+z)=a^2$	$x=\frac{a^2}{\sqrt{(a^2+b^2+c^2)}}$.
$y(x+y+z)=b^2$	$y=\frac{b^2}{\sqrt{(a^2+b^2+c^2)}}$.
$z(x+y+z)=c^2$	$z=\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}$.

Given Equations.	Answers.
30. $xyz = 231$,	$x = 3$.
$xyw = 420$,	$y = 7$.
$yzw = 1540$,	$z = 11$.
$xzw = 660$,	$w = 20$.

31. Shew that the following equations are not sufficient to determine the unknown quantities contained in them, either from being *incongruous*, or *not independent*.

1. $3x - 2y = 14$.
   $6x - 4y = 28$.

2. $x + y = 7$.
   $3x + 3y = 30$.

3. $6x + 9y = 27$.
   $8x + 12y = 36$.

4. $x + y = 7$.
   $3x - y = 1$.
   $x + 2y = 10$.

5. $x + 3y + 4z = 9$.
   $3x - 2y + 17z = 25$.
   $x + 14y - z = 11$.

6. $2x + 3y - z = 16$.
   $5x - 2y + 7z = 21$.
   $3x - 5y + 8z = 12$.

7. $3x - 2y + 5z = 14$.
   $2x + y - 8z = 10$.
   $8x - 3y + 2z = 38$.

CHAP. VIII.—QUADRATIC EQUATIONS.

QUADRATICS CONTAINING ONE UNKNOWN QUANTITY.

ART. 139. DEF. I. A *quadratic* equation, or an equation of the *second degree*, is one in which the unknown quantity is in the second power.

140. II. A quadratic which contains the second power only of the unknown quantity, is called a *pure* quadratic; while one that contains both its second and first powers, is called a *compound* or *adfected* quadratic. Thus, a pure quadratic is of the form $ax^2 = \pm b$; while a compound quadratic is of the form $ax^2 \pm bx = \pm c$.

141. *Schol.* It is usual to regard as a compound quadratic, any equation which contains two such powers of the unknown quantity, that the index of one of them is double that of the other; that is to say, any equation of the form $ax^{2n} \pm bx^n = \pm c$.*

142. III. That quantity which is independent of the unknown, is called the *absolute* term of the equation: thus, in the above equations, c is the absolute term.

* A simple equation is often defined to be one which contains only one power of the unknown quantity; and hence those equations which contain more than one power are called compound. In this view, all pure equations of the second, third, &c. degrees, are considered simple equations, and solved as such. We have, however, considered them as belonging more properly to another class of equations.

143. PROB. I. *To solve a pure quadratic.* Simplify the equation by applying Art. 127, and extract the square root of both members.

For, the general form being

$$ax^2 = \pm c,$$

where c represents the aggregate of all the known terms, we have by division,

$$x^2 = \pm\frac{c}{a} \text{ and } x = \pm\sqrt{\pm\frac{c}{a}}.$$

EXAMPLE.

Given, $\sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{x+2}{x-2}} = 4$, to find the value of x.

Squaring both sides we have

$$\frac{x-2}{x+2} + \frac{x+2}{x-2} + 2 = 16;$$

clearing of fractions and transposing, there is obtained

$$12x^2 = 64;$$

whence

$$x = \pm\frac{8}{\sqrt{12}} = \pm\frac{4}{\sqrt{3}} = \pm\frac{4}{3}\sqrt{3}.$$

144. II. *To solve a compound quadratic.*

RULE I.

(1) Simplify the equation, by applying Art. 127; (2) divide both members by the coefficient of the higher power of the unknown quantity; (3) add to both sides the square of half the coefficient of the lower power of the unknown quantity; (4) extract the square root of both sides; (5) and, from the resulting simple equation, determine the values of the unknown quantity.

The reason of this rule may be explained as follows: dividing both members of the general equation

$$ax^2 \pm bx = \pm c,^*$$

by a, and putting, for brevity,

$$\frac{b}{a} = p, \quad \frac{c}{a} = q,$$

we obtain

$$x^2 \pm px = \pm q.$$

Now, the terms of the left member are plainly the two first terms of the square of a binomial, whose first term is x (Art. 97); and since px is *twice* the product of the two terms of the binomial, the second term must be $\frac{1}{2}p$: the square of this is $\frac{1}{4}p^2$, which, being added to both sides, gives

$$x^2 \pm px + \tfrac{1}{4}p^2 = \pm q + \tfrac{1}{4}p^2.$$

The left member is now a complete square; therefore, by evolution, and Art 96, we have

$$x \pm \tfrac{1}{2}p = \pm\sqrt{(\pm q + \tfrac{1}{4}p^2)};$$

and by transposition,

$$x = \mp\tfrac{1}{2}p \pm \sqrt{(\pm q + \tfrac{1}{4}p^2.)}\dagger$$

Since $\frac{1}{2}p$, and the quantity under the radical sign, are

* The equation $ax^2 \pm bx = \pm c$, is no less general than $\pm ax^2 \pm bx = \pm c$; because, if the first term be negative, it can be rendered positive by changing the signs of all the terms.

† This formula, expressed in words, gives the following rule:

Having prepared the equation as in (1) and (2) of the rule in the text; to the half coefficient of the lower power of the unknown, with its sign changed, annex by the signs $\pm$ the square root of the sum of the absolute term, and the square of the half coefficient.

Thus, if it be required to find x from the equation $x^2 - 10x = 24$, we have $p = 10$ and $q = 24$, and therefore $x = 5 \pm \sqrt{(24+25)} = 5 \pm \sqrt{49} = 5 \pm 7 = 12$, or -2.

both affected by the double sign, it is plain that the above general formula includes the following cases:

1. When $x^2+px=+q$, $x=\begin{cases}-\frac{1}{2}p+\sqrt{(q+\frac{1}{4}p^2)},\\ \text{or}\\ -\frac{1}{2}p-\sqrt{(q+\frac{1}{4}p^2)}.\end{cases}$

2. When $x^2-px=+q$, $x=\begin{cases}\frac{1}{2}p+\sqrt{(q+\frac{1}{4}p^2)},\\ \text{or}\\ \frac{1}{2}p-\sqrt{(q+\frac{1}{4}p^2)}.\end{cases}$

3. When $x^2+px=-q$, $x=\begin{cases}-\frac{1}{2}p+\sqrt{(-q+\frac{1}{4}p^2)}.\\ \text{or}\\ -\frac{1}{2}p-\sqrt{(-q+\frac{1}{4}p^2)}.\end{cases}$

4. When $x^2-px=-q$, $x=\begin{cases}\frac{1}{2}p+\sqrt{(-q+\frac{1}{4}p^2)},\\ \text{or}\\ \frac{1}{2}p-\sqrt{(-q+\frac{1}{4}p^2)}.\end{cases}$

In the third and fourth of these general values, if $\frac{1}{4}p^2 < q$, the quantity within the vinculum will be negative, and therefore (Art. 96) the values of x will be impossible, or imaginary. The first and second forms will always give real values.

EXAMPLE.

Given,

$$\frac{5x^2}{12}+\frac{2}{3}x+6\tfrac{1}{4}=\tfrac{8}{5}x+20-\frac{10x}{3},$$

to find the value of x.

Here, applying parts (1) and (2) of the rule, we find

$$x^2+8x=33;$$

then (3) the square will be completed by adding to both sides the square of the half coefficient of x, that i 16: we thus get

$$x^2+8x+16=33+16,$$

or

$$x^2+8x+16=49;$$

hence (4),

$$x+4=\pm 7$$

and (5),

$$x = \pm 7 - 4;$$

therefore,

$$x = +7 - 4 = 3,$$

and

$$x = -7 - 4 = -11.$$

Applying the rule in the note, we have
$x = -4 \pm \sqrt{(33+16)} = -4 \pm \sqrt{49} = -4 \pm 7 = 3$, or -11.
Verifying this equation, we find

$$9 + 24 = 33,$$

and

$$121 - 88 = 33.$$

RULE II.

145. (1) Simplify the equation, by applying Art. 127; (2) multiply both members by four times the coefficient of the higher power of the unknown quantity; (3) add to both members the square of the coefficient of the lower power of the unknown quantity, in the original equation; (4) extract the square root of both sides; (5) and, from the resulting simple equation, find the values of the unknown quantity.

The reason of this rule may be shown as follows: resuming the general equation

$$ax^2 \pm bx = \pm c,$$

and multiplying both members by $4a$, we obtain

$$4a^2x^2 \pm 4abx = \pm 4ac.$$

Now it is plain, as in the last Art., that if b^2 be added to both sides, the left will be a complete square: we thus get

$$4a^2x^2 \pm 4abx + b^2 = \pm 4ac + b^2;$$

and by evolution,

$$2ax \pm b = \pm\sqrt{(\pm 4ac + b^2)};$$

whence, by transposition and division,

$$x = \frac{\mp b \pm \sqrt{(\pm 4ac + b^2)}}{2a}.*$$

---

* This method of solving quadratics is taken from the "Bija Ganita," a Hindoo Treatise on Algebra, of the date of about the beginning of the thirteenth century, and made known in Europe, through a translation from a Persian copy, by Mr. Edward Strachey, of the Bengal Civil Service. An account of this curious work, and of another treatise on collateral subjects, called "The Leelawuttee," will be found in Dr. Hutton's Mathematical Tracts, vol. II, pp. 151–179.

Subsequently, Dr. Thomson, having never seen or heard of the Hindoo mode of solution, published, in a Belfast periodical, the following rule, which, though the demonstration of it is the same as that in the text, is yet, in a practical point of view, quite different from the Hindoo rule, as it gives the values of x without any intermediate work, such as completing the square, extracting the root, &c. It was not till several years after that Dr. Thomson became acquainted with the Hindoo method.

"Reduce the given equation to the form $ax^2 + bx = c$: then, to find x, to the coefficient of the second term with its sign changed, annex, by the double sign $\pm$, the square root of the quantity obtained, by adding together the square of that coefficient and four times the product of the absolute term and the coefficient of the first term; and divide the whole expression thus obtained by twice this last-mentioned coefficient."

Applying this rule to the equation $3x^2 + 5x = 42$, we have at once

$$x = \frac{-5 \pm \sqrt{(25 + 504)}}{6} = \frac{-5 \pm \sqrt{529}}{6} = \frac{-5 \pm 23}{6} =$$

3, or $-4\frac{2}{3}$.

The second rule may also be immediately derived from the first. Putting the original equation under the form

$$x^2 \pm \frac{b}{a}x = \pm\frac{c}{a},$$

and completing the square, by adding to both sides the square of

EXAMPLE.

Given, $x^2+\frac{2}{3}x-3=25\frac{1}{3}$, to find the values of x.

Simplifying part (1), we have

$$3x^2+2x=85\ ;$$

then (2) multiplying by 4×3, or 12, we find

$$36x^2+24x=1020,$$

and (3)

$$36x^2+24x+4=1024\ ;$$

hence, by evolution (4)

$$6x+2=\pm 32\ ;$$

therefore

$$6x=30, \text{ or } -34,$$

and

$$x=5, \text{ or } -5\tfrac{2}{3}.$$

Applying the Rule in the Note, we should have

$$x=\frac{-2\pm\sqrt{(1020+4)}}{6}=\frac{-2\pm 32}{6}=5, \text{ or } -5\tfrac{2}{3}.$$

---

the half coefficient of x, that is $\frac{b}{2a}$, we obtain

$$x^2\pm\frac{b}{a}x+\frac{b^2}{4a^2}=\pm\frac{c}{a}+\frac{b^2}{4a^2}\ ;$$

hence, by evolution,

$$x\pm\frac{b}{2a}=\pm\sqrt{(\pm\frac{c}{a}+\frac{b^2}{4a^2})}.$$

Adding the fractions under the radical sign, extracting the root of the denominator, and then transposing, we find

$$x=\frac{\mp b\pm\sqrt{(\pm 4ac+b^2)}}{2a}$$

The following method of deriving the same formula, will be a useful exercise in managing surds. In the original equation $ax^2\pm bx=\pm c$, it is plain that $ax^2=(x\sqrt{a})^2$; so that the first term

146. By employing the first of the two preceding Rules, fractions will be introduced into the operation, whenever, in equations reduced to the form $x^2 \pm px = \pm q$, the coefficient p or $\frac{b}{a}$ is odd or fractional. In such cases the second Rule should be used, as fractions will not occur till the last. The first Rule is preferable when, in equations of the form $ax^2 \pm bx = \pm c$, a is a large number, as the operation by the second Rule is then tedious and troublesome.

MISCELLANEOUS EXAMPLES.

1. Given, $6x - 30 = 3x^2$, to find the values of x.

By transposition and division,

$$x^2 - 2x = -10;$$

therefore by Rule I.

$$x^2 - 2x + 1 = -9.$$

---

of the binomial is $x\sqrt{a}$: also that $bx = (x\sqrt{a} \times \frac{b}{2\sqrt{a}}) \times 2$; so that the second term of the binomial is $\frac{b}{2\sqrt{a}}$: hence, $\frac{b^2}{4a}$ will complete the square; we therefore have

$$(x\sqrt{a})^2 \pm 2x\sqrt{a}\,\frac{b}{2\sqrt{a}} + \frac{b^2}{4a} = \pm c + \frac{b^2}{4a};$$

hence, by evolution,

$$x\sqrt{a} \pm \frac{b}{2\sqrt{a}} = \pm\sqrt{(\pm c + \frac{b^2}{4a})}.$$

Transposing, and dividing by $\sqrt{a}$, we find

$$x = \mp\frac{b}{2a} \pm \sqrt{(\pm\,\frac{c}{a} + \frac{b^2}{4a^2})};$$

and by reduction,

$$x = \frac{\mp b \pm \sqrt{(\pm 4ac + b^2)}}{2a}$$

the same as by the two former methods.

By evolution,

$$x-1 = \pm\sqrt{(-9)};$$

whence

$$x = 1 \pm 3\sqrt{-1};$$

both of which values are imaginary, as is obvious from the general values 3 and 4, Art. 144.

2. Given, $5x^2+4x = 25$; required x.

Here Rule II. is to be preferred; therefore,

$$100x^2+80x+16 = 516.$$

Evolving, transposing, and dividing, we have

$$x = \frac{-4 \pm \sqrt{516}}{10} = 1.87516\ldots \text{ or } -2.67156\ldots$$

3. Given, $\sqrt{(x+21)}+\sqrt[4]{(x+21)} = 12$, to find the values of x.

Completing the square, we have

$$\sqrt{(x+21)}+\sqrt[4]{(x+21)}+\tfrac{1}{4} = 12\tfrac{1}{4} = \tfrac{49}{4};$$

by evolution,

$$\sqrt[4]{(x+21)}+\tfrac{1}{2} = \pm\tfrac{7}{2},$$

whence

$$\sqrt[4]{(x+21)} = 3, \text{ or } -4;$$

whence

$$x = 60, \text{ or } 235.$$

4. Given, $3x-\dfrac{3x-10}{9-2x} = 2+\dfrac{6x^2-40}{2x-1}$, to determine x.

Multiplying by $2x-1$, and simplifying, and again by $9-2x$, and simplifying, we find

$$8x^2-124x = -368;$$

by division,

$$x^2-\frac{31}{2}x = -46.$$

Appyling Rule I. which is in this case the simpler, we find

$$x^2-\frac{31}{2}x+\frac{961}{16}=-46+\frac{961}{16}=\frac{225}{16};$$

by evolution,

$$x-\frac{31}{4}=\pm\frac{15}{4};$$

whence

$$x=4, \text{ or } 11\tfrac{1}{2}.$$

5. Given, $\frac{x+\sqrt{(x^2-9)}}{x-\sqrt{(x^2-9)}}=(x-2)^2$, to find the values of x.

Here (Art. 117),

$$\frac{(x+\sqrt{x^2-9})^2}{9}=(x-2)^2;$$

by evolution,

$$\frac{x+\sqrt{(x^2-9)}}{3}=\pm(x-2).$$

Taking the right member positive, and making various reductions, we find

$$x^2-8x=-15;$$

which gives $x=5$, or 3.

Again, taking the right member negative, and reducing as before, there results

$$x=\frac{8\pm\sqrt{11\sqrt{-1}}}{5};$$

these two values are imaginary; the two others are real.

6. Given, $2x^2+3x-5\sqrt{(2x^2+3x+9)}+3=0$, to find x. Here the quantity under the radical sign exceeds the quantity outside of it, by 6; therefore, if 6 be added to both members, the first unknown term will be the square of the second: adding, we have

$$2x^2+3x+9-5\sqrt{(2x^2+3x+9)}=6$$

solving this equation by the usual method, and then squaring, we have

$$2x^2+3x+9=36, \text{ or } 1.$$

The first of these gives

$$x=3, \text{ or } -4\tfrac{1}{2};$$

and the second,

$$x=\frac{-3\pm\sqrt{(-55)}}{4}.$$

7. Given, $x^4-2x^3+x=132$.

It is plain that the equation will not be changed, by adding x^2 to the left member, and then subtracting it; this gives

$$x^4-2x^3+x^2-x^2+x=132,$$

or

$$(x^2-x)^2-(x^2-x)=132.$$

Solving this equation, we find, for the two values,

$$x^2-x=12, \text{ and } x^2-x=-11:$$

the first equation gives

$$x=4, \text{ or } -3;$$

the second,

$$x=\frac{1\pm\sqrt{43}\sqrt{(-1)}}{2}.$$

8. Given, $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$, to find x.

By transposition,

$$\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}-\frac{x^2}{a}=-\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}};$$

squaring, transposing, and reducing by Art. 75, we find

$$x^2-\frac{2x^2}{a}\sqrt{\left(\frac{x^4-a^4}{x^2}\right)}+\frac{x^4-a^4}{a^2}=0,$$

that is,

$$x^2-\frac{2x}{a}\sqrt{(x^4-a^4)}+\frac{x^4-a^4}{a^2}=0.$$

which is a complete square. Evolving, transposing, clearing of fractions, and squaring, we find

$$a^2x^2 = x^4 - a^4;$$

whence, $$x^4 - a^2x^2 = a^4.$$

Solving this quadratic, we find

$$2x^2 = a^2 \pm a^2\sqrt{5} = a^2(1 \pm \sqrt{5}),$$

and

$$x = \pm a\sqrt{\left(\frac{1 \pm \sqrt{5}}{2}\right)}.$$

9. Given, $\frac{a-x}{x} + \frac{x}{a-x} = \frac{b}{c}$; required the values of x.

Multiplying all the terms by $\frac{a-x}{x}$, and transposing, we find

$$\left(\frac{a-x}{x}\right)^2 - \frac{b}{c}\cdot\frac{a-x}{x} = -1.$$

Completing the square, and extracting the root, we have

$$\frac{a}{x} - 1 = \frac{b}{2c} \pm \frac{\sqrt{(b^2+4c^2)}}{2c};$$

therefore,

$$x = \frac{2ac}{2c+b \pm \sqrt{(b^2+4c^2)}}.$$

10. Given, $\frac{49x^2}{4} + \frac{48}{x^2} - 49 = 9 + \frac{6}{x}$; required x.

Here, it is easy to see that the square will be completed on both sides, by adding $\frac{1}{x^2}$: this gives

$$\frac{49x^2}{4} - 49 + \frac{49}{x^2} = 9 + \frac{6}{x} + \frac{1}{x^2};$$

evolving and reducing, we find

$$7x^2 - 6x = 16, \text{ and } 7x^2 + 6x = 12:$$

the solution of these two equations gives

$$x = 2, \text{ or } -\tfrac{8}{7};$$

and

$$x = \frac{-3 \pm \sqrt{93}}{7} = 0.949\ldots, \text{ or } -1.8062\ldots$$

EXERCISES.

I. *Pure quadratics.*

Given Equations.	Answers.
1. $\frac{3x^2}{5} - 1 = 8\frac{3}{5}$	$x = \pm 4$.
2. $\frac{5x^2}{2} + x^2 = 126$	$x = \pm 6$.
3. $x^2 + ab = 5x^2$	$x = \pm \frac{1}{2}\sqrt{ab}$.
4. $x^2 - 17 = 130 - 2x^2$	$x = \pm 7$.
5. $3x^2 - 4 = 28 + x^2$	$x = \pm 4$.
6. $x^2 - 6x + 9 = 25 - 30x + 9x^2$	$x = 2$.
7. $y^2 = 9a^6$	$y = \pm 3a^3$.
8. $x + \sqrt{(a^2 + x^2)} = \frac{2a^2}{\sqrt{(a^2 + x^2)}}$	$x = \pm \frac{a}{\sqrt{3}}$.
9. $\frac{a - \sqrt{(a^2 - x^2)}}{a + \sqrt{(a^2 - x^2)}} = b$	$x = \pm \frac{2a\sqrt{b}}{1 + b}$.
10. $\frac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} = b$	$x = \frac{2ab}{b^2 + 1}$.
11. $x^2 - 2x + 1 = 9$	$x = 4$, or -2.
12. $\frac{\sqrt{(4x+1)} + \sqrt{4x}}{\sqrt{(4x+1)} - \sqrt{4x}} = 9$	$x = \frac{4}{9}$.

II. *Compound quadratics.*

Given Equations.	Answers.
1. $x^2 - 6x = 55$	$x = 11$, or -5.
2. $5x^2 - 10x = 75$	$x = 5$, or -3.
3. $2x^2 + 32x + 110 = 0$	$x = -11$, or -5.

Given Equations.	Answers.
4. $2x^2-3x=54$	$x=6$, or $-4\frac{1}{2}$.
5. $5x^2-4x=156$	$x=6$, or $-5\frac{1}{5}$.
6. $3x^2-5x=2$	$x=2$, or $-\frac{1}{3}$.
7. $x^2-13x+40=0$	$x=5$, or 8.
8. $6x^2+5x=4$	$x=\frac{1}{2}$, or $-\frac{4}{3}$.
9. $x^2+16x+48=0$	$x=-12$, or -4.
10. $x^2-20x+84=0$	$x=14$, or 6.
11. $4x^2-20x=336$	$x=12$, or -7.
12. $2x^2+3x-2=0$	$x=\frac{1}{2}$, or -2.
13. $x^2+12x+35=0$	$x=-5$, or -7.
14. $3x^2-48x=-180$	$x=10$, or 6.
15. $y^2-y+3=45$	$y=7$, or -6.
16. $6x^2-5x+1=0$	$x=\frac{1}{2}$, or $\frac{1}{3}$.
17. $\frac{x^2}{3}-\frac{x}{10}+\frac{1}{6}=\frac{1}{3}$	$x=\frac{1}{2}$, or $-\frac{1}{5}$.
18. $5x^2+35x=600$	$x=8$, or -15.
19. $x^2-8x=14$	$x=4\pm\sqrt{30}$.
20. $\frac{2x^2}{3}+3\frac{1}{2}=\frac{x}{2}+8$	$x=3$, or $-2\frac{1}{4}$.
21. $\frac{x^2}{10}+350-12x=0$	$x=70$, or 50.
22. $x^2-\frac{3}{10}x=\frac{1}{10}$	$x=\frac{1}{2}$, or $-\frac{1}{5}$.
23. $x-\frac{x^3-8}{x^2+5}=2$	$x=2$, or $\frac{1}{2}$.
24. $17x^2+19x=1848$	$x=9\frac{15}{17}$, or 11.
25. $x=1+\frac{110}{x}$	$x=11$, or -10.
26. $2x^2+26x+40=-20$	$x=-10$, or -3.
27. $6y^2-33y+45=0$	$y=3$, or $2\frac{1}{2}$.
28. $3x-\frac{1}{4}x^2=10$	$x=6\pm2\sqrt{(-1)}$.

Given Equations.	Answers.
29. $2x^2+15=3x$	$x=\frac{3\pm\sqrt{-111}}{4}$.
30. $x^2-x=11342$	$x=107$, or -106.
31. $x^2-6x+19=13$	$x=4.732..$, or $1.268...$
32. $4x-\frac{36-x}{x}-46=0$	$x=12$, or $-\frac{3}{4}$.
33. $\frac{4x}{9}+\frac{x-5}{x+3}=\frac{4x+7}{19}$	$x=3$, or -8.7.
34. $2x^2-31x+14=-1$	$x=15$, or $\frac{1}{2}$.
35. $\frac{4x^2}{5}+8x-50\frac{1}{4}=429\frac{3}{4}$	$x=20$, or -30.
36. $\frac{x}{x+60}=\frac{7}{3x-5}$	$x=14$, or -10.
37. $\frac{16}{x}-\frac{100-9x}{4x^2}-3=0$	$x=4$, or $2\frac{1}{12}$.
38. $\frac{7}{x+1}+\frac{2}{x}-5=0$	$x=1.14..$, or -0.34.
39. $(x+6)^2-(x+5)^2=(x-2)^2$	$x=7$, or -1.
40. $\frac{4}{x-3}-\frac{3}{x+5}=\frac{1}{18}$	$x=8\pm\sqrt{601}$.
41. $x+2\sqrt{x}=24$	$x=16$, or 36.
42. $2x^4-7x^2-99=0$	$x=3$, or $\sqrt{\frac{-11}{2}}$.
43. $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$	$x=2$, or -3.
44. $\frac{\sqrt{4x+2}}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$	$x=4$, or $7\frac{1}{9}$.
45. $\sqrt{x^3}-2\sqrt{x}=x$	$x=4$, or 1.
46. $x^2+13+\sqrt{(x^2+13)}=56$ ..	$x=\pm 6$, or $\pm\sqrt{51}$.
47. $\frac{8}{x^{\frac{1}{2}}}+2-\frac{17}{x^{\frac{1}{4}}}=0$	$x=4$, or $\sqrt[3]{\frac{1}{4}}$.
48. $\sqrt[5]{x}+\sqrt[5]{x^2}=6$	$x=32$, or -243.

Given Equations.	Answers.
49. $(x-5)^3-3(x-5)^{\frac{1}{2}}=40$	$x=9$, or $5+(-5)^{\frac{1}{2}}$.
50. $x^{2n}-4x^n=10$	$x=\sqrt[n]{(2\pm\sqrt{14})}$.
51. $x^6+3x^3=810$	$x=3$, or $\sqrt[3]{-30}$.
52. $\frac{x}{2}-\frac{\sqrt{x}}{3}-22\frac{1}{6}=0$	$x=49$, or $5\frac{6}{9}\frac{1}{9}$.
53. $5x^6-90x^3-270=945$	$x=3$, or $\sqrt[3]{-9}$.
54. $x^4-74x^2+1225=0$	$x=\pm 5$, or ± 7.
55. $x^6-x^3=a$	$x=\sqrt[3]{\frac{1\pm\sqrt{(4a+1)}}{2}}$.
56. $x^2-2px-q=0$	$x=p\pm\sqrt{(p^2+q)}$.
57. $\frac{x}{a}+\frac{a}{x}-\frac{2}{a}=0$	$x=1\pm\sqrt{(1-a^2)}$.
58. $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}$	$x=\frac{5}{4}a$, or $\frac{1}{2}a$.
59. $\frac{a^2-b^2}{c}=2ax-cx^2$	$x=\frac{a\pm b}{c}$.
60. $x^6-7x^3=8$	$x=2$, or -1.
61. $\frac{\frac{3\sqrt{x}}{5}-2}{x-5}=\frac{1}{20}$	$x=49$, or 25.
62. $x^2-2x+6\sqrt{(x^2-2x+5)}=11$ ..	$x=1$, or $1\pm 2\sqrt{15}$.
63. $\sqrt{(x+5)}=\frac{12}{\sqrt{(x+12)}}$	$x=4$, or -21.
64. $3x^2-x^{\frac{5}{3}}=3104x^{\frac{1}{3}}$	$x=64$, or $(-\frac{97}{3})^6$.
65. $3x^2+8b^2-24ax=17b^2$	$x=4a\pm\sqrt{(16a^2+3b^2)}$.
66. $\sqrt{x}+\sqrt{(a-x)}=\sqrt{b}$	$x=\frac{(a\pm\sqrt{2ab-b^2})}{2}$.
67. $\frac{x}{a}+bx^{-1}=c$	$x=\frac{ac\pm\sqrt{(a^2c^2-4ab)}}{2}$.
68. $(a+b)x^2-cx=\frac{ac}{a+b}$	$x=\frac{c\pm\sqrt{(c^2+4ac)}}{2(a+b)}$.

Given Equations. ANSWERS.

69. $ax+2\sqrt{(n^2x+nax^2)}=n(3x-1)$ .. $x=\frac{n}{n-a}$ or $\frac{n}{9n-a}$.

70. $\frac{x^2}{8a}+\frac{2x}{3}=\sqrt{\left(\frac{x^3}{3a}+\frac{x^2}{4}\right)}-\frac{a}{2}$ .. $x=6a$, or $-\frac{2}{3}a$.

71. $\sqrt{(a+x)}+\sqrt{(a-x)}=\frac{12a}{5\sqrt{(a+x)}}$ .. $x=\frac{4}{5}a$, or $\frac{5}{3}a$.

72. $mqx^2-mnx+pqx-np=0$ ..... $x=\frac{n}{q}$, or $-\frac{p}{m}$.

73. $\sqrt{(2x+7)}+\sqrt{(3x-18)}=\sqrt{(7x+1)}$ .. $x=9$, or $-3\frac{5}{7}$.

74. $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$ ..... $x=4$, or 1.

75. $ax^{-\frac{2}{3}}-bx^{-\frac{1}{3}}=c$ ..... $x=\left(\frac{-b\pm\sqrt{(4ac+b^2)}}{2c}\right)^3$.

76. $\sqrt[3]{(x^3-a^3)}=x-b$ ..... $x=\frac{1}{2}b\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

77. $\sqrt{x^5}-3x=40x^{-\frac{1}{2}}$ ..... $x=4$, or $(-5)^{\frac{2}{3}}$.

78. $x+\sqrt{x}:x-\sqrt{x}::3\sqrt{x}+6:2\sqrt{x}$ .... $x=9$, or 4.

79. $1+\frac{1}{\sqrt{(x+1)}}=\sqrt{\frac{x^3}{x^2-1}}$ ..... $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

80. $\sqrt{(x-2)^2-x^2}-(x-2)^2-x=90$.

Ans. $x=6$, or -1, or $\frac{5\pm3\sqrt{-3}}{2}$.

81. $x+5=\sqrt{(x+5)}+6$ ..... $x=4$, or -1.

82. $\sqrt{(x+60)}+\sqrt{(x^2+9)}=\frac{2\sqrt{(x^3+60x^2+9x+540)}+89}{\sqrt{(x+60)}+\sqrt{(x^2+9)}}$ .. $x=4$, or -5.

83. $\frac{x}{\sqrt{x}+\sqrt{(a-x)}}+\frac{x}{\sqrt{x}-\sqrt{(a-x)}}=\frac{b}{\sqrt{x}}$

Ans. $x=\frac{b\pm\sqrt{(b^2-2ab)}}{2}$.

84. $\frac{a-\sqrt{(2ax-x^2)}}{a+\sqrt{(2ax-x^2)}}=\frac{x}{a-x}$ ..... $x=a$, or $\frac{1}{3}a$.

QUADRATICS CONTAINING TWO UNKNOWN QUANTITIES.

147. The following form is the most general under which quadratics with two unknown quantities can appear.

$$ax^2+bxy+cy^2+dx+ey=f,$$
$$a'x+b'xy+c'y^2+d'x+e'y=f'.$$

Now, if we eliminate x^2 from these equations, and deduce an expression for x, that expression will contain y^2 and y; and, therefore, when this expression is substituted in either of the original equations, the resulting equation will contain y^4, y^2, and y: hence, the solution of the two given equations implies the solution of an equation of the fourth degree. But when the first powers of the unknowns are wanting, and the equations have the form

$$ax^2+bxy+cy^2=d,$$
$$a'x^2+b'xy+c'y^2=d',*$$

they can be solved as equations of the second degree. Besides these, there are equations of various forms which can be solved by applying the general rules of elimination already laid down. There is not, however, any general method of solution, which will apply to all equations of this class: their solution is effected by adopting various artifices of analysis, which will be best illustrated by particular examples.

* Such equations are called *homogeneous*, each term on the left side being of two dimensions with respect to the unknown quantities, (Art, 17). It is worthy of remark, that if equations be given homogeneous, every step in the process of solution, if it be correct, will give a homogeneous equation; and thus a simple test is afforded of the accuracy of the various operations.

EXAMPLES.

1. Equations of the form

$$ax^2+bxy+cy^2=d,$$
$$a'x^2+b'xy+c'y^2=d',$$

in which the unknowns are similarly involved, will be best resolved by putting $x=zy$, deducing two expressions for y^2, equating them, and clearing of fractions: in this way we get

$$a'dz^2+b'dz+c'd=ad'z^2+bd'z+cd',$$

or

$$(a'd-ad')z^2+(b'd-bd')z=cd'-c'd.$$

This quadratic will give z; y and x can then be determined.

2. Equations of the form

$$x^n+y^n=a, \quad (1)$$
$$xy=b, \quad (2)$$

may be solved as follows:

From the square of (1), subtract four times the nth power of (2), extract the square root of the remainder, and there is obtained

$$x^n-y^n=\sqrt{(a^2-4b^n)};$$

combining this with equation (1), we find

$$x=[\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-4b^n)}]^{\frac{1}{n}},$$
$$y=[\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-4b^n)}]^{\frac{1}{n}}.$$

3. Equations such as

$$x+y=a, \quad (1)$$
$$x^3+y^3=b, \quad (2)$$

may be resolved by either of the following methods:

From the cube of (1) subtract (2), and there remains

$$3x^2y+3xy^2=a^3-b,$$

or

$$3(x+y)xy=3axy=a^3-b;$$

hence,

$$x = \frac{a^3 - b}{3ay};$$

and by substitution, transposition, and division by $3a$,

$$y^2 - ay = \frac{b - a^3}{3a};$$

therefore (Art. 144),

$$y = \frac{a}{2} \pm \sqrt{\frac{4b - a^3}{12a}}$$

hence,

$$x = \frac{a}{2} \mp \sqrt{\frac{4b - a^3}{12a}}.$$

Otherwise put $x = v + z$, $y = v - z$; then, substituting these in (2), we find

$$z^2 = \frac{b - 2v^3}{6v},$$

and

$$z = \pm \sqrt{\frac{b - 2v^3}{6v}};$$

therefore,

$$x = v \pm \sqrt{\frac{b - 2v^3}{6v}},\ y = v \mp \sqrt{\frac{b - 2v^3}{6v}}.$$

Now,

$$x + y = 2v = a;$$

therefore,

$$v = \frac{a}{2};$$

hence we have

$$x = \frac{a}{2} \pm \sqrt{\frac{b - \frac{a^3}{4}}{3a}} = \frac{a}{2} \pm \sqrt{\frac{4b - a^3}{12a}},$$

and

$$y = \frac{a}{2} \mp \sqrt{\frac{4b - a^3}{12a}}.$$

This mode of solution is not applicable to powers above the fifth, as a cubic equation would be produced.

4. Given, $x+y : x-y :: 13 : 5$,

$$y^2+x=25.$$

By a well known property of proportionals,

$$2x : 2y :: 18 : 8,$$

or

$$x : y :: 9 : 4;$$

therefore,

$$4x=9y, \text{ and } x=\frac{9y}{4}.$$

Substituting this in the second equation, we have

$$y^2+\frac{9y}{4}=25;$$

whence (Art. 144),

$$y=4, \text{ or } -6\tfrac{1}{4}; \; x=9, \text{ or } -14\tfrac{1}{16}.$$

5. Given, $$\frac{10x+y}{xy}=3, \quad (1)$$

$$9x-9y=-18. \quad (2)$$

From (1),

$$10x+y=3xy,$$

and from (2),

$$y=x+2;$$

then, by substitution and transposition, we have

$$3x^2-5x=2;$$

whence (Art. 145),

$$x=2, \text{ or } -\tfrac{1}{3}; \; y=4, \text{ or } \tfrac{5}{3}.$$

6. Given, $$x^4-2x^2y+y^2=49, \quad (1)$$

$$x^4-2x^2y^2+y^4-x^2+y^2=20. \quad (2)$$

Equation (2) may be put under the form

$$(x^2-y^2)^2-(x^2-y^2)=20.$$

Solving this quadratic, we find

$$x^2-y^2 = 5, \text{ or } -4;$$

also from (1),

$$x^2-y = \pm 7:$$

by subtraction,

$$y^2-y = 2, \text{ or } -12;\ 11, \text{ or } -3.$$

Solving this quadratic for these four values, we find

$$y = 2, \text{ or } -1; \qquad x = \pm 3, \text{ or } \pm\sqrt{6}.$$

$$y = \frac{1 \pm \sqrt{-47}}{2}; \quad x = \pm\sqrt{\frac{-13 \pm \sqrt{-47}}{2}}.$$

$$y = \frac{1 \pm 3\sqrt{5}}{2}; \quad x = \pm\sqrt{\frac{15 \pm 3\sqrt{5}}{2}}.$$

$$y = \frac{1 \pm \sqrt{-11}}{2}; \quad x = \pm\sqrt{\frac{-13 \pm \sqrt{-11}}{2}}.$$

7. Given, $x^2+y^2+x+y = 18, \quad (1)$

$$2xy = 12. \quad (2)$$

Adding the two equations, and solving the quadratic which results, we find

$$x+y = 5, \text{ or } -6; \quad (3)$$

hence, from (1)

$$x^2+y^2 = 13, \text{ or } 24;$$

from this taking (2), and evolving, we have

$$x-y = \pm 1, \text{ or } \pm 2\sqrt{3}; \quad (4)$$

combining (4) with (3), we find

$$x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3};$$

and

$$y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}.$$

8. Given, $xy+xy^2 = 12,$

$$x+xy^3 = 18.$$

Equating the expressions for x, deduced from these two

equations, dividing by $\frac{6}{1+y}$, and then reducing and transposing, we find

$$2y^2-5y=-2;$$

whence,

$$y=2, \text{ or } \tfrac{1}{2};$$

and therefore,

$$x=2, \text{ or } 16.$$

9. Given, $x^{\frac{3}{2}}+y^{\frac{3}{2}}=3x$, (1)

$$x^{\frac{1}{2}}+y^{\frac{3}{2}}=x. \quad (2)$$

From the square of (2) subtract (1), and in the remainder substitute for $y^{\frac{3}{2}}$ an expression deduced from (2), and by transposing and dividing, we shall obtain

$$x-x^{\frac{1}{2}}=2;$$

which gives

$$x=4, \text{ or } 1;$$

and therefore,

$$y=8, \text{ or } 0.$$

10. Given, $\sqrt{y}+\sqrt{x} : \sqrt{y}-\sqrt{x} :: \sqrt{x}+2 : 1$, (1)

$$\frac{\sqrt{y}+2}{\sqrt{x}}-1=\frac{3\sqrt{x}+1+\frac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}}, \quad (2)$$

From (1)

$$\sqrt{y} : \sqrt{x} :: \sqrt{x}+3 : \sqrt{x}+1;$$

therefore,

$$\sqrt{xy}+\sqrt{y}=x+3\sqrt{x}; \quad (3)$$

and from (2),

$$y+2\sqrt{y}-\sqrt{xy}=3x+\sqrt{x}+\sqrt{y};$$

by transposition,

$$y+\sqrt{y}-\sqrt{xy}=3x+\sqrt{x}. \quad (4)$$

The addition of (3) and (4) gives a quadratic; solving it, we have

$$\sqrt{y}+1 = \pm(2\sqrt{x}+1);$$

whence,

$$\sqrt{y} = 2\sqrt{x}, \text{ and } \sqrt{y} = -2\sqrt{x}-2.$$

From the first of these values, and from (3), we find

$$x = \sqrt{x};$$

whence, $x = 1,$

and therefore $y = 4.$

From the second value, and (1), we obtain

$$-\sqrt{x}-2 : -3\sqrt{x}-2 :: \sqrt{x}+2 : 1,$$

or

$$\sqrt{x}+2 : 3\sqrt{x}+2 :: \sqrt{x}+2 : 1;$$

hence,

$$3\sqrt{x}+2 = 1,$$

and therefore

$$x = \tfrac{1}{9};$$

whence

$$\sqrt{y} = -\tfrac{8}{3},$$

and

$$y = \tfrac{64}{9}.$$

EXERCISES.

I. *Pure quadratics.*

Given Equations.	Answers.
1. $x+y = 3x-3y,$	$x = 4.$
$x^3-y^3 = 56.$	$y = 2.$
2. $xy+y^2 = 126,$	$x = \pm 15.$
$5y = 2x.$	$y = \pm\ 6.$
3. $x^2+y^2 : x^2-y^2 :: 17 : 8,$	$x = 5.$
$xy^2 = 45.$	$y = 3.$
4. $x-y : \sqrt{x}-\sqrt{y} :: 8 : 1,$	$x = 25,$ or $9.$
$\sqrt{xy} = 15.$	$y = 9$ or $25.$

Given Equations.	Answers.
5. $\frac{x^2+xy+y^2}{x+y}=7,$	$x=6.$
$\frac{x^2-xy+y^2}{x-y}=9.$	$y=3.$

II. *Compound quadratics.*

Given Equations.	Answers.
1. $4xy=96-x^2y^2,$	$x=4$, or 2, or $3\pm\sqrt{21}.$
$x+y=6.$	$y=2$, or 4, or $3\mp\sqrt{21}.$
2. $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9},$	$x=5$, or $\frac{17}{10}.$
$x-y=2.$	$y=3$, or $-\frac{5}{10}.$
3. $x-\frac{x-y}{2}=4,$	$x=2$, or 5.
$y-\frac{x+3y}{x+2}=1.$	$y=6$, or 3.
4. $x+4y=14,$	$x=2$. or $-46.$
$4x-2y+y^2=11.$	$y=3$, or 15.
5. $x+y=a,$	$x=\frac{a\pm\sqrt{(a^2-4b)}}{2}.$
$xy=b.$	$y=\frac{a\mp\sqrt{(a^2-4b)}}{2}.$
6. $x+y=a,$	$x=\frac{a\pm\sqrt{(2c-a^2)}}{2}.$
$x^2+y^2=c.$	$y=\frac{a\mp\sqrt{(2c-a^2)}}{2}.$
7. $6x^2-5xy+2y^2=12,$	$y=\pm 3$, or $\pm\frac{6}{\sqrt{31}}.$
$3x^2+2xy-3y^2=-3.$	$x=\pm 2\mp\frac{5}{\sqrt{31}}.$

Given Equations.	Answers.
8. $x^2+xy=12$,	$x=\pm 3$, or $\pm\frac{8}{\sqrt{6}}$.
$xy-2y^2=1$.	$y=\pm 1$, or $\pm\frac{1}{\sqrt{6}}$.
9. $(x+y)x=144$,	$x=\pm 9$, or $8\sqrt{2}$.
$(x-y)y=14$.	$y=\pm 7$, or $\sqrt{2}$.
10. $x^3+y^3=axy$,	$x=b+b\sqrt{\frac{a-2b}{a+6b}}$.
$x+y=2b$.	$y=b-b\sqrt{\frac{a-2b}{a+6b}}$.
11. $x^4+y^4=97$,	$x=3$, or 2.
$xy=6$.	$y=2$, or 3.
12. $x+y=20$,	$x=11$, or 9.
$x^3+y^3=2060$.	$y=9$, or 11.
13. $x^2+y^2=306$,	$x=9$, or 15.
$x+y=24$.	$y=15$, or 9.
14. $x-2\sqrt{xy}+y-\sqrt{x}+\sqrt{y}=0$,	$x=9$, or $6\frac{1}{4}$.
$\sqrt{x}+\sqrt{y}=5$.	$y=4$, or $6\frac{1}{4}$.
15. $(x^2+y^2)(x^3+y^3)=280$,	$x=3$.
$x+y=4$.	$y=1$.
16. $x-y=8$,	$x=11$, or -52.
$x^4-y^4=14560$.	$y=3$, or -60.
17. $x-y=2$,	$x=5$, or 1.7.
$\frac{x^2}{y^2}=\frac{85-36x}{9y}$.	$y=3$, or -3.
18. $x^3+y^3=152$,	$x=5$, or 3.
$x^2+xy^2=120$.	$y=3$, or 5.

Given Equations.	Answers.
19. $x+y-\frac{\sqrt{(x+y)}}{\sqrt{(x-y)}}=\frac{6}{x-y}$,	$x=5$, or $\sqrt{\frac{45}{2}}$.
$x^2+y^2=41$,	$y=4$, or $\sqrt{\frac{37}{2}}$.
20. $x+y+z=21$,	$x=12$, or 3.
$x^2+y^2+z^2=189$,	$y=6$.
$y^2=xz$.	$z=3$, or 12.
21. $x^5+y^5=17050$,	$x=3$.
$x+y=10$.	$y=7$.
22. $x^{x+y}=y^{4a}$,	$x=\frac{4a+1\mp\sqrt{(8a+1)}}{2}$.
$y^{x+y}=x^a$.	$y=\frac{-1\pm\sqrt{(8a+1)}}{2}$.
23. $2x^2-3xy+3y^2=5$,	$x=\pm 2$, or $\pm\frac{23}{58}\sqrt{29}$.
$3x^2+xy-2y^2=12$.	$y=\pm 1$, or $\pm\frac{9}{58}\sqrt{29}$.
24. $\frac{\sqrt{(y^2+1)}+1}{y}=\frac{\sqrt{(x+9)}+3}{\sqrt{x}}$.	$x=\frac{57\pm\sqrt{105}}{2}$.
$x(y+1)^2=36(y^3+\frac{16}{9})$	$y=\frac{3\pm\sqrt{105}}{6}$.

CHAP. IX.—SOLUTION OF PROBLEMS.

ART. 148. The object of an algebraical problem is to find the value of some unknown quantity, by means of the relations which it bears to others that are known; and since equations have the same object (Art. 122), it is plain that problems may be expressed by equations. In order to effect this, the data of the problem, or the known quantities, and the required or unknown quantities, are denoted as is explained in Art. 122; and the same operations are then performed on these quantities, as are described by the conditions of the problem. It is, of course, assumed, that these conditions must be such as to establish a mutual dependence between the known and unknown quantities, and as to be expressible by means of the ordinary operations of Algebra. With respect to the number of equations which the conditions will give, it is sufficient to refer to Art. 129.

The chief difficulty in the solution of a problem consists in adapting the operations to be performed to the conditions given by the problem; that is, in the reduction of the problem to algebraical language. For this, however, no fixed Rule can be laid down; and it must therefore be learned by practice.*

* Considerable difficulty is often felt in the first attempts to translate the conditions of a question into algebraic language. To remove this difficulty, the Miscellaneous Exercises, pp. 56, 57. have been given, chiefly by way of preparation for this chapter.

When it happens, from the nature of the question, that fractional parts of an unknown are to be taken, it will be better to represent the unknown by such a multiple of x, y, &c. as will avoid fractions. The answers which are obtained should be verified by trying if they will fulfil the conditions of the problem.

MISCELLANEOUS PROBLEMS.

1. What number is that, whose half exceeds its third part as much as 7 exceeds the number itself?

Let x represent the number.

Then the excess of the half above the third of the number, is $\frac{x}{2} - \frac{x}{3}$; and the excess of 7 above the number itself is $7 - x$: therefore, by the conditions of the question, we have

$$\frac{x}{2} - \frac{x}{3} = 7 - x.$$

which gives

$$x = 6.$$

To avoid fractions, we might take $6x$ for the number; we should then have

$$3x - 2x = 7 - 6x;$$

whence,

$$x = 1, \text{ and } 6x = 6;$$

the number required.

2. Of a bequest of £4755, B got three times as much as A, C as much as A and B together, and D as much as C and B. Required the share of each.

Here, the other shares being expressed in terms of A's,

let x represent A's share; then B's is $3x$, C's is $x+3x$, and D's is $x+3x+3x$: then, by the question, we have

$$x+3x+x+3x+x+3x+3x=4755;$$

which gives A's share £317; B's, £951; C's, £1268; D's, £2219.

3. If a steam-carriage, starting from London, travel 26 miles per hour; and one from Liverpool, 24 miles per hour; in what time will they meet, and what distance will each have travelled, the distance by the railroad being supposed to be 200 miles.

Let the number of hours required be x. Then $26x$ and $24x$ are the respective distances travelled; therefore, according to the data,

$$26x+24x=200;$$

whence,

$$x=4.$$

They meet in 4 hours; and the distances passed over are 104 miles, and 96 miles.

4. If A can do a piece of work in a days, and B in b days; in how many days will they do the work, if they both work at it together?

Let x be the number of days. Then, in one day, A can do $\frac{1}{a}$ of the work; and B, $\frac{1}{b}$: therefore, in x days, they can do $\frac{x}{a}$, and $\frac{x}{b}$, of the work.

Therefore, representing the whole work by 1, we have the equation

$$\frac{x}{a}+\frac{x}{b}=1;$$

whence, $$x = \frac{ab}{a+b};$$

from which particular solutions may be deduced.

5. How many gallons of spirits, at 9 shillings per gallon, must be mixed with 20, at 13 shillings, so that the mixture may be worth 10 shillings per gallon?

Let x be the quantity at 9 shillings. Then $9x$ shillings is its price.

Also, the whole quantity is $x+20$, and its price is $10(x+20)$ shillings; and 260 shillings is the price of 20 at 13 shillings. Then

$$9x+260 = 10x+200;$$

therefore,

$$x = 60;$$

and the whole quantity is 80.

6. A person being asked the hour, answered that it was between 5 and 6 o'clock, and that the hour and minute hands were exactly together. What is the time?

Let x = the time past 5. Then, as the minute hand makes 12 revolutions for one of the hour hand, we have

$$12 : 1 :: 5+x : x;$$

whence, the time is easily found to be $27'\ 16\frac{4}{11}''$ past 5 o'clock.

7. A general, ranging a division of his army in the form of a solid square, finds he has 34 men to spare; but, increasing the side by one man, he wants 59 to fill up the square. How many men had he?

Let x and $x+1$ be the numbers of men in the sides of the two squares: then, x^2+34, and $x^2+2x+1-59$, are the numbers of men in each. Hence, we easily find the number of men to be 2150; and the numbers in the two sides, 46 and 47.

8. There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 9 be subtracted from twice the number, the digits will be inverted. Required the number.

Let the digits be x and y: then the number is $10x+y$, and the number inverted is $10y+x$. Therefore, by the question

$$10x+y=4(x+y),$$

and

$$2(10x+y)-9=10y+x;$$

from which two equations we find the number to be 36.

9. A merchant sold a quantity of brandy for £39, and gained as much per cent. as the brandy cost him. Required the price of the brandy.

Let $x=$ the price of the brandy; then £39—x is the gain. Also,

$$39-x : x :: x : 100;$$

whence,

$$x^2=3900-100x,$$

or

$$x^2+100x=3900.$$

Solving this quadratic, we find

$$x=30, \text{ or } -130;$$

the price is therefore £30.*

* This negative value shows (Art. 5), that the question must be modified. It will be as follows:

"A merchant sold a quantity of brandy, by which he *lost* £39 *more* than the first cost, and found that his loss was as much per cent as the brandy cost him. Required the price."

The equation will now be

$$x^2=3900+100x;$$

and the values of x will be 130, and —30: the price is therefore £130. The value —30 shows that *this* question must be modified; when the conditions are changed, it will, of course, be as in the text.

10. A rectangular bowling-green is of a certain size; and if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required its length and breadth.

Let $x =$ the length, and $y =$ the breadth; then the area $= xy$. Also,

$$(x+4)\times(y+5) = xy+116,$$

and

$$(x+5)\times(y+4) = xy+113;$$

from which we find

$$x = 12,\ y = 9.$$

11. A vintner sold 7 dozen of sherry, and 12 dozen of claret, for £50; and finds that he has sold 3 dozen more of sherry for £10, than he has of claret for £6. Required the price of each.

Let $x =$ the price of a dozen of sherry, in pounds; then $\frac{10}{x} =$ number of dozen of sherry for £10, and $\frac{10}{x} - 3 =$ number of dozen of claret for £6: therefore, $6 \div (\frac{10}{x} - 3) =$ the price of a dozen of claret. Hence, $7x + \frac{72x}{10-3x} = 50$, From this we find that the sherry was £2, and the claret £3.

This might also be solved as follows: Let the prices be x and y: then, by the question

$$7x+12y = 50,$$

and

$$\frac{10}{x} - \frac{6}{y} = 3.$$

Deducing two expressions for x from these two equations, equating them, and solving the resulting quadratic, we find $y = £3$; and therefore, from the first equation, $x = £2$.

12. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel, in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x and y be the number of yards in the circumferences of the wheels; then $\frac{120}{x}$ and $\frac{120}{x+1}$, $\frac{120}{y}$ and $\frac{120}{y+1}$, express the number of revolutions of the wheels: we have therefore $\frac{120}{x} = \frac{120}{y} - 6$, and $\frac{120}{x+1} = \frac{120}{y+1} - 4$. From the first of these, we find $xy = 20x - 20y$; from the second, $xy = 29x - 1 - 31y$. Equating these two, deducing an expression for x, and substituting it in the equation $xy = 20(x - y)$, a quadratic is obtained. Solving it, we find $y = 4$, and therefore $x = 5$.

EXERCISES.*

1. What number is that whose half and third part together are equal to 20? *Ans.* 24.

2. What is that number whose quadruple exceeds its half by 14? *Ans.* 4.

3. What number is that, the double of which increased by b, is equal to a? *Ans.* $\frac{1}{2}(a - b)$.

* The questions which are solved in general terms, should be illustrated by examples with particular numbers.

4. What number is that from whose double if a be taken, the remainder will be b? *Ans.* $\frac{1}{2}(a+b)$.

5. What number is that, $\frac{1}{a}$ and $\frac{1}{b}$ of which together are equal to c? *Ans.* $\frac{abc}{a+b}$.

6. My coat and hat together are worth £3 10s.; but the latter is only one-fourth the value of the former.

Ans. The hat is worth 14s. the coat worth £2 16s.

7. Divide 56 into two parts, having the ratio of 11 to 17.

Ans. 34, 22.

8. Divide £150 between two persons, giving the one £14 more than the other.

Ans. The shares are £82 and £68.

9. What is that number, from whose treble if 4 be subtracted, three-fifths of the remainder will be 12? *Ans.* 8.

10. Divide £300 among three persons, giving the second £9 more than the third, and the first £15 more than the second. *Ans.* The shares are £89, £98, £113.

Solve this question in general terms, by putting $300=c$, $9=a$, $15=b$.

Ans. The shares are $\frac{c-2a-b}{3}$, $\frac{c+a-b}{3}$, $\frac{c+a+2b}{3}$.

By means of these solve such questions as the following:

1. A cask which holds 100 gallons is filled with a mixture of wine, water, and spirits; there are 15 gallons more of wine than of water, and 25 gallons of spirits more than of wine.

Ans. Water 15, wine 30, spirits 55.

2. Three workmen, A, B, C, were cutting a drain 159 perches long, of which B cut 15 perches more than A, and C 24 perches more than B. How much did each cut?

Ans. A cut 35, B 50, and C 74 perches.

11. Required a number, the half of which is as much less than 100, as its double exceeds 100. *Ans.* 80.

The following is the general statement of this question:

Find a quantity x, such that mx may be as much less than a, as nx is greater than b.

Ans. $x = \frac{a+b}{m+n}$.

12. A post is one-third in mud, one-fourth in water, and 10 feet above water. Required its length.

Ans. 24 feet.

13. A fish was caught whose tail weighed 9lbs., its head as much as its tail and half its body, and its body as much as its head and tail. Required its whole weight.

Ans. 72lbs.

14. If two persons start at the same time from London and Glasgow, and travel, the former 36 miles per day, and the latter 30; in how many days will they meet, and how far from London, the whole distance being 396 miles?

Ans. In 6 days, and 216 miles from London.

15. The sum of the three sides of a triangle is 75; the base is 11 feet longer than one side, and 16 feet longer than the other. Required the three sides.

Ans. 34, 23, 18.

16. Find two numbers such that their sum shall be a, and the one m times as great as the other.

Ans. $\frac{a}{m+1}$ and $\frac{ma}{m+1}$.

17. Divide a into three such parts that the second may be m times, and the third n times as great as the first.

Ans. $\frac{a}{1+m+n}$, $\frac{ma}{1+m+n}$, $\frac{na}{1+m+n}$.

This is the general statement of such questions as the following:

In a company of 45 persons, there are three times as many

English, and five times as many Irish, as Scotch. Required the number of each nation *Ans.* 5 Scotch, 15 English, 25 Irish.

18. At a certain election a persons voted, and the successful candidate had a majority of b persons. How many voted for each? *Ans.* $\frac{1}{2}(a-b)$ and $\frac{1}{2}(a+b)$.

19. A man, at the time he was married, was three times as old as his wife; but after they had lived together 15 years, he was only twice as old. What were their ages on their wedding day?

Ans. The woman's 15, the man's 45.

20. A cistern which holds 820 gallons, is filled in 20 minutes by three pipes, of which one conveys 10 gallons per minute more, and the other 5 gallons per minute less than the third. How much does each pipe convey in a minute? *Ans.* 22, 7, 12.

21. Having bought a certain quantity of brandy at 19 shillings a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings a gallon; I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each?

Ans. 34 of brandy, and 43 of rum.

22. A man and his wife usually drank out a vessel of beer in 12 days; but when the man was from home, the vessel lasted the woman 30 days. In how many days would the man alone drink it out? *Ans.* 20 days.

23. It is required to divide 34 into two such parts, that the difference between the greater and 18 shall be to the difference between 18 and the less, as 2 is to 3.

Ans. 22, 12.

24. The perimeter of a triangle is 43; but one side is 10 inches longer than the other, which is three inches longer than the base. Required the sides.

Ans. The sides are 12, 9, 22.

25. If A can perform a piece of work in a days, B in b days, and C in c days; in how many days will they have finished the work, if they all work at it together?

Ans. In $\frac{abc}{ab+ac+bc}$ days.

26. If A can perform a piece of work in a days, B in b days, C in c days, and D in d days; in how many days will they finish it, if they all work at it together?

Ans. In $\frac{abcd}{abc+abd+bdc+adc}$ days.

27. At what time, between 11 and 12 o'clock, are the hour and minute hands of a clock exactly together?

Ans. $5\frac{5}{11}$ minutes before twelve.

28. A hare is 50 leaps before a greyhound, and takes four leaps to the greyhound's three; but two of the greyhound's are as much as three of the hare's. How many leaps must the greyhound take to catch the hare?

Ans. 300.

29. Find two numbers in the proportion of 3 to 4, whose sum is to the sum of their squares as 7 to 50.

Ans. 6 and 8.

30. Find two numbers in the proportion of 9 to 7, so that the square of their sum and the cube of their difference shall be equal. *Ans.* 288 and 224.

31. In 1830, A's age was 50, and B's 35. Required the date at which A's age is double of B's.

Ans. 1810.*

32. A father's age is now treble of his son's; what are their present ages, the age of the father 7 years ago having been five times that of the son? *Ans.* 14 and 42.

To find a general solution for this question, let $3=m$, $5=n$, and $7=a$: then $x=\frac{a(n-1)}{n-m}$.

33. A merchant has two kinds of wine; one costs 36 pence per quart, and the other 20 pence: he wishes to make a mixture of 50 quarts, which shall sell for 30 pence. How much of each sort must he take?

Ans. $18\frac{3}{4}$ quarts of the one, and $31\frac{1}{4}$ of the other.

34. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces; how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and one of tin $4\frac{1}{4}$ ounces?

Ans. 420 copper, 85 tin.

35. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value; but being detected, the pieces were found to be worth only 8 guineas. Required their original value.

Ans. £10 8s.

36. One person says to another, "Give me £100, and

* If this question were enunciated as follows, "In 1830, A's age was 50, and B's 35; when will A's age be double of B's;" the answer, $x=-20$, would show that the date is *before* 1830, and therefore that the words "when will A's age be" are incorrect, and must be changed for the words "Required the date at which A's age is." *See* Art. 5 *and* 6.

then I shall have as much as you." "No," answered the other; "but do you give me £100, and I shall have twice as much as you." How much has each?

Ans. The one £500, and the other £700.

37. A wine merchant sold at one time 20 dozen of port wine, and 30 of sherry, and for the whole received £120; and again, at the same prices, 30 dozen of port, and 25 of sherry, and for the whole got £140. Required the price of a dozen of each sort. *Ans.* £3 and £2.

38. Required a fraction, to the numerator of which if 4 be added, the value is $\frac{1}{2}$; but if 7 be added to the denominator, the value is $\frac{1}{3}$. *Ans.* $\frac{5}{18}$.

Or for a general solution:

Required a fraction, which, if a be added to its numerator, will become b; but if c be added to its denominator, will become d.

Ans. $\frac{ad+bdc}{a+dc}$.

39. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts used was 100. How many were there of each sort, the guinea being 21, and the moidore 27 shillings? *Ans.* 50 of each.

40. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either, alone, drink a barrel?

Ans. The man in $21\frac{3}{7}$ days, the woman in 50 days.

41. With 28 bushels of barley, at 2*s.* 4*d.* per bushel, a farmer would mix rye at 3*s.*, and wheat at 4*s*; so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* per bushel. How much rye and wheat must he take? *Ans.* 20 of rye, and 52 of wheat.

42. A countryman being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled three feathers out of the tail of each turkey, and one out of the tail of each goose; and, upon counting them, found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought ten geese, and sold fifteen turkeys, by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first. *Ans.* 45 turkeys, and 60 geese.

43. A number consisting of two digits, when divided by 4, gives a certain quotient, and a remainder of 3; when divided by 9, gives another quotient, and a remainder of 8. Now, the value of the digit on the left hand is equal to the quotient which was got when the number was divided by 9; and the other digit is $\frac{1}{17}$th of the quotient got when the number was divided by 4. Required the number.

Ans. 71.

44. If A and B can do a piece of work, in 8 days, A and C in 9 days, and B and C in 10 days; in how many days can each alone perform the same work?

Ans. In $14\frac{14}{49}$, $17\frac{25}{41}$, $23\frac{7}{31}$ days.

45. A boy spends his money in oranges. If he had received 5 more for his money, they would have cost a halfpenny less each; but if 5 less, a halfpenny more each. How much did he spend? *Ans.* 2*s.* 6*d.*

46. There is a field in the form of a rectangular parallelogram, whose length exceeds its breadth by 16 yards; and it contains 960 square yards. Required the length and breadth. *Ans* 40 and 24.

47. There are three numbers, the difference of whose differences is 5; their sum is 44, and continued product 1950. Find the numbers. *Ans.* 25, 13, and 6.

48. Having sold goods for £24, I gained as much per cent as the whole cost me. Required the cost.

Ans. £20, or — 120.

49. There are two square buildings, that are paved with stones a foot square each: the side of one building exceeds that of the other by 12 feet; and both their pavements taken together, contain 2120 stones. What are their lengths separately? *Ans.* 26 and 38.

50. A person bought some sheep for £72, and found, that if he had bought 6 more for the same money, he would have paid £1 less for each. Required the number bought, and the price of each.

Ans. The number was 18, and the price of each £4.

51. The difference between the hypotenuse and base of a right-angled triangle is 6; and the difference between the hypotenuse and perpendicular is 3. Required the sides. *Ans.* 9, 12, 15.

52. There is a field in the form of a rectangle, whose length is to its breadth as 6 to 5; one-sixth of the whole area being planted, there remain for ploughing 625 square yards. Required the dimensions of the field.

Ans. 30 by 25.

53. Two messengers were dispatched at the same time to a place 90 miles distant; the former of whom, by riding one mile an hour faster than the other, arrived at the end of his journey an hour before the other. Required the rate of each per hour.

Ans. 10 and 9 miles per hour.

54. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has 4 men more

in a side than the other. Required the number in each side. *Ans.* 21 and 25.

55. A company at a tavern had £8 15*s.* to pay; but before the bill was paid, two of them went away, and in consequence, those who remained had each 10*s.* more to pay. How many persons were in the company at first?

Ans. 7 persons.

56. Find two numbers whose product is 300, and such that if 10 be added to the less, and 8 subtracted from the greater, the product of the sum and remainder shall also be 300. *Ans.* 20 and 15.

57. A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle; at the end of the time he receives a pence. How many days did he work, and how many was he idle?

Ans. $\frac{nq+a}{p+q}$, and $\frac{np-a}{p+q}$.

58. A poulterer bought 15 ducks and 12 turkeys for 5 guineas, and had two ducks more for 18*s.* than he had of turkeys for 20 shillings. What was the price of each?

Ans. 3*s.* and 5*s.*

59. The sum of the numerator and denominator of a fraction is equal to 6 times the fraction; and the sum of the numerator and denominator of its reciprocal is equal to $1\frac{1}{2}$ times its reciprocal. What is the fraction?

Ans. $\frac{1}{2}$.

60. It is required to find the three sides of a right-angled triangle, from the following data: the number of square feet in the area is equal to the number of feet in the hypotenuse, together with the sum of the feet in the other two sides; and the square described upon the hypotenuse

s less than the square described upon the sum of the two sides, by half the product of the base and area.

Ans. Hypotenuse 10, the sides 6 and 8.

61. A man travelled 105 miles, and found that, if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the same journey. How many miles did he go per hour? *Ans.* 7.

62. A farmer bought two flocks of sheep for £65 13*s*. one containing 5 more than the other: each sheep cost as many shillings as there were sheep in the flock. Required the number in each. *Ans.* 23 and 28.

63. The joint stock of two partners, A and B, was £416; A's money was in trade 9 months, and B's 6 months: when they shared stock and gain, A received £228, and B £252. Required each man's stock.

Ans. The stocks are £192 and £224.

64. Two persons, A and B, travelled on the same road and at the same rate to London. At the 50th milestone from London A overtook a flock of geese, which travelled at the rate of 3 miles in 2 hours, and 2 hours afterwards he met a stage waggon which travelled at the rate of 9 miles in 4 hours. B overtook the flock of geese at the 45th milestone from London, and met the stage waggon 40 minutes before he came to the 31st milestone. Where was B when A reached London?

Ans. 25 miles from London.

65. In comparing the rates of a watch and a clock, it was observed, on one morning, when it was 12 by the clock that the watch was at 59 minutes 49 seconds past 11; and, two mornings after, when it was 9 by the clock, the watch was at 59 minutes 58 seconds past 8. The clock is known to gain $\frac{1}{10}$ second in 24 hours. Find the gaining rate of the watch. *Ans.* 4.9 seconds in 24 hours.

66. If A and B together can perform a piece of work in a days, A and C together in b days, and B and C together in c days, find the time in which each can perform the work separately.

Ans. A in $\frac{2abc}{ac+bc-ab}$ days; B in $\frac{2abc}{ab+bc-ac}$ days; C in $\frac{2abc}{ab+ac-bc}$ days.

67. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished 4 times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men, when their number was found to be increased in the ratio of 17 to 16. How many companies were in the regiment?

Ans. 12 companies.

68. The product of the sum and difference of the hypotenuse and a side of a right-angled triangle is equal to 2; and 4 times the sum of the squares of the hypotenuse and this side is equal to 5 times the sum of these two lines. Find the three sides of the triangle.

Ans. $1\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{2}\sqrt{5}$

CHAP. X.—METHOD OF INDETERMINATE COEFFICIENTS.

ART. 149. If it were required to resolve the fraction $\frac{a}{a+bx}$ into an infinite series, without actually dividing, we at once see (Art. 81) that the series will consist of the powers of x multiplied by certain coefficients, the indices of the powers of x, beginning at zero and increasing by a unit in each successive term. That such development is possible, will also appear from Art. 92, since the given expression is the same as $a(a+bx)^{-1}$. It is not, however, obvious what the coefficients will be; but it is plain that they are independent of x, and are expressed in terms of a and b. Let us then assume the coefficients to be A, B, C, D, &c. and we shall have $\frac{a}{a+bx} = 1+Ax+Bx^2+Cx^3+Dx^4$, &c.,

an *identical* equation, the assumed series being merely the supposed result of the operation indicated by the left member. To determine these coefficients, multiply both members by the denominator, and arrange the terms by the powers of x; we thus obtain

$$\begin{array}{l|l|l|l|l} a = a+Aa & x+Ba & x^2+Ca & x^3+Da & x^4 \text{ \&c. } * \\ \quad +b & \quad +Ab & \quad +Bb & \quad +Cb & \end{array}$$

* That the left member is of the same form as the right, will be plain when it is written thus, $a+0x+0x^2+0x^3$, &c.

Now, in order that this equation may be satisfied by its members being rendered identical, the terms containing x must vanish. This will take place, whatever be the value of x, when the coefficients of its powers are equal to zero; that is, when

$Aa+b=0$, from which we deduce $A=-\frac{b}{a}$;

$Ba+Ab=0$ ..... $B=-A\frac{b}{a}=\frac{b^2}{a^2}$;

$Ca+Bb=0$ ..... $C=-B\frac{b}{a}=-\frac{b^3}{a^3}$;

$Da+Cb=0$ ..... $D=-C\frac{b}{a}=\frac{b^4}{a^4}$; &c.

The series will therefore be

$$\frac{a}{a+bx}=1-\frac{b}{a}x+\frac{b^2}{a^2}x^2-\frac{b^3}{a^3}x^3+\frac{b^4}{a^4}x^4 \text{ \&c.};$$

the same as would be obtained by actual division, which in this case would be more direct.

This indirect method of determining the development of a series, is called the method of *assumed* or *indeterminate coefficients*. It is applied to the same purposes as the direct methods in Art. 81, 91, 99, and to various others in the higher mathematics.*

The general principle of the method consists in assuming unknown coefficients, and then deducing two expressions or series of the same form, which, by the nature of the process by which they are obtained, are to be identical with one another. A series of equations, giving

* *Undetermined coefficients*, or *coefficients to be determined*, would be a better name than *indeterminate*; as that word has, in other parts of Algebra, a different signification.

the values of these coefficients, is then obtained by equating the corresponding terms, or their coefficients, and thus rendering the expressions identical.

150. If the series $A+Bx+Cx^2+Dx^3$ &c. be equal to the series $A'+B'x+C'x^2+D'x^3$ &c. for all values of x, then will the coefficients of x in the two series be respectively equal to one another. For, equating these two series, and transposing all the terms to the right member, we have

$$0 = A'-A+(B'-B)x+(C'-C)x^2+(D'-D)x^3 \text{ &c.};$$

which cannot be true universally, unless

$$A'-A = 0,\ B'-B = 0,\ C'-C = 0,\ D'-D = 0.$$

Hence, $A' = A,\ B' = B,\ C' = C,\ D' = D$ &c.

151. A series with indeterminate coefficients, is generally assumed to proceed according to the ascending integral and positive powers of x, beginning with x^0; but in many series this is not the case: the error in our assumption is then shown, either by some impossible result, or by those terms which do not exist in the actual series becoming zero in the course of the calculation. Thus, if it be required to develope $\frac{1}{3x-x^2}$, assuming the same series as before, we should have

$$0 = -1+3Ax+(3B-A)x^2+(3C-B)x^3+(3D-C)x^4 \text{ &c.}$$

which would give $0 = -1$, $A = \frac{1}{3}$ &c. The first of these is absurd, and from the others nothing can be determined; the form of the assumed series must, therefore, be modified.

Putting $\frac{1}{3x-x^2}$ under the form $\frac{1}{x} \times \frac{1}{3-x}$, let the assumed

series express the latter factor; then

$$\frac{1}{x} \times \frac{1}{3-x} = \frac{1}{x}(A+Bx+Cx^2+Dx^3) \text{ &c.}$$

from which we find, by the usual process,

$$A = \tfrac{1}{3},\ B = \tfrac{1}{9},\ C = \tfrac{1}{27},\ D = \tfrac{1}{81},\ \text{&c.}$$

then

$$\frac{1}{3x-x^2} = \frac{1}{x}(\tfrac{1}{3}+\tfrac{1}{9}x+\tfrac{1}{27}x^2+\tfrac{1}{81}x^3 \text{ &c.})$$

or

$$\frac{1}{3x-x^2} = \tfrac{1}{3}x^{-1}+\tfrac{1}{9}x^0+\tfrac{1}{27}x^1+\tfrac{1}{81}x^2 \text{ &c.}$$

Again, if we assumed

$$\frac{1-x^2}{1+x^2-x^4} = A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6, \text{ &c.}$$

we should find the true series to be

$$1-2x^2+3x^4-5x^6+8x^8 \text{ &c.}$$

the coefficients B, D, F, &c. of the odd powers of x becoming zero. The series above assumed will, therefore, answer the purpose as well as if we had assumed

$$\frac{1-x^2}{1+x^2-x^4} = A+Cx^2+Ex^4+Gx^6 \text{ &c.}$$

in which the terms that contain the odd powers are wanting.

152. The following miscellaneous examples will serve to illustrate some of the applications of this method:

Ex. 1. Required the development of $\sqrt{(a^2+x^2)}$.

Assume $\sqrt{(a^2+x^2)} = A+Bx+Cx^2+Dx^3+$ &c.

Squaring both sides so as to obtain quantities of the same form, we find

$$a^2+x^2=A^2+2ABx+2AC \left|\begin{array}{l} \\ +B^2 \end{array}\right| x^2+2AD \left|\begin{array}{l} \\ +2BC \end{array}\right| x^3+2AE \left|\begin{array}{l} +2BD \\ +C^2 \end{array}\right| x^4+ \&c.$$

Hence, equating the corresponding coefficients, we get

$A^2=a^2$, $2AB=0$, $2AC+B^2=1$, $2AD+2BC=0$, &c.

The first of these gives $A=a$, the second $B=0$; the others give by substitution the respective values, $C=\frac{1}{2a}$, $D=0$, $E=-\frac{1}{8a^3}$ &c. the series required therefore is

$$\sqrt{(a^2+x^2)}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3} \&c.$$ See Example 1, Art. 99.

2. Let $x=ay+by^2+cy^3+$ &c.; required the value of y in terms of x.

Assume $y=Ax+Bx^2+Cx^3+$ &c.

$$\left.\begin{array}{rl} \text{then } ay= & aAx+aBx^2+aCx^3+ \&c. \\ by^2= & \quad bA^2x^2+2bABx^3+ \&c. \\ cy^3= & \quad\quad cA^3x^3+ \&c. \\ \&c. & \quad\quad \&c. \\ -x= & -x \end{array}\right\} =0;$$

hence

$$aA-1=0, \text{ or } A=\frac{1}{a}; \quad aB+bA^2=0, \text{or } +B=-\frac{b}{a^3}$$

$$aC+2bAB+cA^3=0, \text{or } C=\frac{2b^2-ac}{a^5} \&c.$$

We have therefore

$$y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{(2b^2-ac)x^3}{a^5} \&c.$$

3. Let it be required to resolve the fraction $\frac{6x^3-2x-6}{(x^2+1)(x+2)(x-3)}$ into partial fractions, whose denominators shall be the factors of the given denominator.

Assume $\frac{6x^3-2x-6}{(x^2+1)(x+2)(x-3)} = \frac{A+Bx}{x^2+1} + \frac{C}{x+2} + \frac{D}{x-3}$.

Adding the partial fractions, and comparing the coefficients of the like powers of x in the numerator of the results, and in that of the given fraction, we find the following series of equations for determining the assumed coefficients:

$$B+C+D=6,\ A-B-3C+2D=0,$$
$$A+6B-C-D=2,\ 6A+3C-2D=6.$$

These equations give $A=1,\ B=1, C=2,\ D=3$; and the partial fractions therefore are

$$\frac{x+1}{x^2+1},\ \frac{2}{x+2},\ \frac{3}{x-3}.$$

4. Resolve $\frac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}$ into its partial fractions.

Assume $\frac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)} = \frac{P}{1+ax} + \frac{Q}{1+bx} + \frac{R}{1+cx}$; then

$A+Bx+Cx^2 = P(1+bx)(1+cx)+Q(1+ax)(1+cx)+R(1+ax)(1+bx)$.

The equation being identical, or true for any value of x, let $1+ax=0$, or $x=-\frac{1}{a}$, then

$$A-\frac{B}{a}+\frac{C}{a^2} = P(1-\frac{b}{a})(1-\frac{c}{a});$$

whence $$P = \frac{Aa^2 - Ba + C}{(a-b)(a-c)}.$$

Similarly if $1+bx=0$, or $x=-\frac{1}{b}$, and $1+cx=0$, or $x=-\frac{1}{c}$, we shall find

$$Q = -\frac{Ab^2 - Bb + C}{(a-b)(b-c)}; \text{ and } R = \frac{Ac^2 - Bc + C}{(a-c)(b-c)}.$$

The three partial fractions will be obtained by dividing these values of P, Q, and R, by $1+ax$, $1+bx$, and $1+cx$, respectively.

The two preceding examples, as well as some of the exercises that follow, will serve to illustrate the nature of an important branch of analysis, the decomposition of rational fractions; it is impossible to enter further on the subject here.

EXERCISES.

1. Convert $\frac{a}{b+cx}$ into a series.

Ans. $\frac{a}{b}\left(1 - \frac{c}{b}x + \frac{c^2}{b^2}x^2 - \frac{c^3}{b^3}x^3 \text{ \&c.}\right)$

2. Convert $\frac{1+2x}{1-x-x^2}$ into a series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6$ &c.

3. Convert $\sqrt{(1-x)}$ into a series.

Ans. $1 - \frac{x}{2} - \frac{x^2}{2.4} - \frac{3x^3}{2.4.6} - \frac{3.5x^4}{2.4.6.8} - \frac{3.5.7x^5}{2.4.6.8.10}$.

4. Convert $\frac{a+bx}{a'+b'x+c'x}$ into a series.

Ans. $\frac{a}{a'} - \left(\frac{b'}{a'}A - \frac{b}{a'}\right)x - \left(\frac{b'}{a'}B - \frac{c'}{a'}A\right)x^2 - \left(\frac{b'}{a'}C + \frac{c'}{a'}B\right)x^3$ &c.

5. Let $x = y - ay^3 + by^5 -$ &c. required the value of y in terms of x.

Ans. $y = x + ax^3 + (3a^2 - b)x^5$ &c.

6. Resolve $\frac{x+1}{x^2-7x+12}$ into partial fractions.

Ans. $\frac{-4}{x-3}$, and $\frac{5}{x-4}$.

7. If $x = y - \frac{y^2}{2} + \frac{y^3}{3} -$ &c. find y in terms of x.

Ans. $y = x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4}$, &c.

8. Find the fractions of which $\frac{x^2}{(x^2-1)(x-2)}$ is the sum.

Ans. $\frac{4}{3(x-2)} - \frac{1}{2(x-1)} + \frac{1}{6(x+1)}$.

9. Let $y^3 - 3y + x = 0$, required the value of y in a series of ascending powers of x.

Ans. $y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} +$ &c.

10. Expand $\frac{3+2x}{5+7x}$ in a series of ascending powers of x.

Ans. $\frac{3}{5} - \frac{11}{5^2}x + \frac{7.11}{5^3}x^2 - \frac{7^2.11}{5^4}x^3 + \frac{7^3.11}{5^5}x^4$ &c.

11. Resolve $\frac{1}{(x+a)(x+b)(x+c)}$ into its partial fractions.

Ans. $\frac{1}{(a-b)(a-c)(x+a)} - \frac{1}{(a-b)(b-c)(x+b)} + \frac{1}{(a-c)(b-c)(x+c)}$.

12. Find the three fractions whose sum is $\frac{1}{x^4-a^4}$.

Ans. $\frac{1}{4a^3(x-a)} - \frac{1}{4a^3(x+a)} - \frac{1}{2a^2(x^2+a^2)}$.

CHAP. XI.—PROPORTION AND PROGRESSION.

ART. 153. DEF. I. *Ratio* is the relation which two quantities of the same kind have to one another, with respect to the number of times the one is contained in the other.

Ratio is expressed by two dots, or by the sign of division: thus, the ratio of a to b is expressed by $a : b$, $a \div b$, or $\frac{a}{b}$.

154. *Cor.* A ratio is not altered by multiplying or dividing its terms by the same quantity (Art. 67).

155. II. The terms of a ratio are called the *antecedent* and *consequent*.

156. III. If the second of four quantities is contained in the first as often as the fourth in the third, the first is said to have to the second *the same ratio* as the third to the fourth.

157. IV. When the second of four quantities is contained in the first oftener than the fourth in the third, the first is said to have to the second *a greater ratio* than the third to the fourth.

Thus, $4 : 2$ is a greater ratio than $4 : 3$, and $8 : 5$ a less ratio than $7 : 4$; because $\frac{12}{6} > \frac{8}{6}$, and $\frac{52}{20} < \frac{55}{20}$.

158. V. The ratio of one quantity to another is said to be *compounded* of two or more ratios, when the number which is the measure (Art. 46) of the ratio in question is equal to the product of the measures of the other ratios. Thus, the ratio of 3 to 108 is compounded of $6 : 3$, $8 : 4$,

15 : 5, 12 : 4, or of any others whose measures are factors of 36. Also, the ratio $ace : bdf$ is compounded of $a : b$, $c : d$, $e : f$.*

159. VI. A compound ratio whose measure is the product of two equal factors, that is, the second power, of the measure of another ratio, is called the *duplicate* of that ratio. In like manner, the ratio of two quantities is said to be the *triplicate* ratio of two other quantities, when its measure is the third power of the measure of the ratio of two other quantities. Thus, 3 : 75 is duplicate of 4 : 20, and 3 : 192 is triplicate of 4 : 16.

160. VII. *Proportion*, or *analogy*, is the equality of ratios, and quantities having the same ratio are called *proportionals*. Thus, if $a : b$ be equal to $c : d$, these quantities constitue a proportion, which is written $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$.†

* For if b be contained x times in a, and d be contained y times in c, and f be contained z times in e, bdf will be contained xyz times in ace. Hence we might define compound ratio thus: In any number of ratios, if the product of the antecedents be taken, as also that of the consequents, the products are said to have to one another a ratio compounded of the other ratios.

† When there are four quantities, such that the difference of the first and second is the same as that of the third and fourth, the quantities are called *equidifferent*. The term *arithmetical* proportion is sometimes applied to such quantities; and for the sake of distinction, proportion as defined above, is called *geometrical* proportion.

It follows, from the definition of equidifferent quantities, that the sum of the means is equal to that of the extremes; for if a, b, c, d, be four such quantities, then $a-b=c-d$; therefore, $a+d=b+c$. Also, if a, b, c, be three equidifferent quantities, then $b-a=c-b$; whence, $b=\frac{a+c}{2}$, that is, the middle term is half the sum of the extremes. This is generally called an *arithmetical mean*.

161. VIII. A *continual proportion*, sometimes called a *geometrical progression*, is a series of terms such that each term bas the same constant ratio to the one next following. Thus, a, ar, ar^2, ar^3, ar^4, ar^5, ... ar^{n-1}, is a series of continual proportionals, in which r is the common ratio, and ar^{n-1} the last term.*

162. *Cor.* If three quantities be continual proportionals, the ratio of the first to the third is duplicate of the ratio of the first to the second; and if there be four quantities, the ratio of the first to the fourth is triplicate of the ratio of the first to the second. For, let there be the two series,

$$ar^2, ar, a;$$

$$ar^3, ar^2, ar, a;$$

then the measure of $ar^2 : a$ is r^2, and of $ar^2 : ar$ is r; also, the measure of $ar^3 : a$ is r^3, and of $ar^3 : ar^2$ is r. Hence, the ratio of the squares is duplicate, and of the cubes triplicate, of the ratio of the first powers.†

163. IX. Three or four quantities are said to be in *harmonical* proportion, when the first has the same ratio to the last that the difference of the first and second has to the difference of the last and last but one. Thus, a, b, c, are in harmonical proportion, when $a : c :: a—b : b—c$; and a, b, c, d, are in harmonical proportion, when $a : d :: a—b : c—d$.

* The terms *arithmetical* and *geometrical*, commonly applied to proportions and progressions, are improper, as they lead to a misconception of the nature and connexion of these series. The terms *equidifferences*, and *continual proportionals*, are much more expressive of their nature.

† The ratio of the square roots is called *sub-duplicate*, and of the cube roots *sub-triplicate* of the ratio of the first powers.

164. X. An *equidifferent series*, sometimes called an *arithmetical progression*, is a series of terms which increase or decrease by the same common difference. Thus, the numbers 20, 17, 14, 11, 8, 5, 2, form an equidifferent series: and also a, $a+d$, $a+2d$, $a+3d$, $a+4d$, ... $a+(n-)d$; in which d is the *common difference*, and $a+(n-1)d$ the last term. If the series were a decreasing one, the last term would be $a-(n-1)d$.

165. *Cor.* In an equidifferent series, the sum of the two extremes is equal to the sum of any two terms equally distant from them, or to twice the middle term, if the number of terms be odd: for, in the above series, if we take the first and fifth terms as extremes, we have $a+(a+4d) = a + d + (a + 3d) = 2(a + 2d)$; and $a + (a + 2d) = 2(a+d)$, or $a+d=\frac{a+(a+2d)}{2}$. *See Note to* Art. 160.

166. THEOR. I. If four quantities be proportionals, their like powers and roots will also be proportionals.

Let $a : b :: c : d$, be a proportion; then (Art. 160) $\frac{a}{b} = \frac{c}{d}$: whence (Art. 31, 67), $\frac{a^n}{b^n} = \frac{c^n}{d^n}$, or (Art. 160) $a^n : b^n :: c^n : d^n$.

167. II. If four quantities be proportionals, the product of the means is equal to the product of the extremes; and if the product of two quantities be equal to the product of two others, the four quantities will form a proportion.

Let the proportion be $a:b::c:d$; then $\frac{a}{b}=\frac{c}{d}$; and clearing of fractions, $ad=bc$. Again, let $mq=np$; divide both sides by nq, then $\frac{m}{n}=\frac{p}{q}$, or $m:n::p:q$.

168. *Cor.* 1. Hence, if the proportion consist of three terms, the product of the extremes is equal to the square of the mean, and conversely. Hence, also, a mean proportional between two quantities is the square root of their product; for in the proportion $a:b::b:c$, $b^2=ac$, and $b=\sqrt{ac}$. This is called a *geometric mean.*

169. *Cor.* 2. Hence, also, any three terms in a proportion being given, we may find the fourth from the equation $ad=bc$.

170. III. If four quantities be proportionals, they are also proportionals when taken alternately or inversely.

Let $a:b::c:d$, or $\frac{a}{b}=\frac{c}{d}$, be the proportion, and let both sides be multiplied by $\frac{b}{c}$; then $\frac{a}{c}=\frac{b}{d}$, and therefore $a:c::b:d$.

Again, let $a:b::c:d$; hence (Art. 167), $ad=bc$, and therefore $b:a::d:c$.

171. IV. In a proportion, the first term is to the second increased or diminished by any multiple of the first, as the third is to the fourth increased or diminished by any multiple of the third.

Let $a:b::c:d$; then inversely (Art. 170), $b:a::d:c$;

hence, $\frac{b}{a} = \frac{d}{c}$, and (Art. 27, 28) $\frac{b}{a} \pm m = \frac{d}{c} \pm m$: whence

$$\frac{b \pm ma}{a} = \frac{d \pm mc}{c},$$

and therefore

$$b \pm ma : a :: d \pm mc : c;$$

and inversely,

$$a : b \pm ma :: c : d \pm mc.$$

172. *Cor.* Hence, $a : c :: b \pm ma : d \pm mc$, therefore $\frac{a}{c} = \frac{b \pm ma}{d \pm mc}$; but $\frac{a}{c} = \frac{b}{d}$, therefore $b : d :: b \pm ma : d \pm mc$; and alternately, $b : b \pm ma :: d : d \pm mc$.

Also, since $\frac{a}{c} = \frac{b+ma}{d+mc}$, and $\frac{a}{c} = \frac{b-ma}{d-mc}$, we have, alternately, $b+ma : b-ma :; d+mc : d-mc$.

Hence, if $m = 1$, $b+a : b-a :: d+c : d-c$; and from the last proportion in Art. 171, $a : b \pm a :: d : d \pm c$, and $b \pm a : a :: d \pm c : c$. Also, $b : b \pm a :: d : d \pm c$, and $b \pm a : b :: d \pm c : d$.

173. V. When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let there be any number of proportionals,

$$a : b :: c : d :: e : f;$$

then $ad = bc$, and $af = be$; also $ab = ba$; therefore,

$$ab+ad+af = ba+bc+be,$$

or

$$a(b+d+f) = b(a+c+e);$$

whence (Art. 167),

$$a : b :: a+c+e : b+d+f.$$

174. VI. If the first and second terms of a pro-

portion be multiplied or divided by any quantity, as also the third and fourth, the quantities which result will be proportional.

Let the proportion be $\frac{a}{b} = \frac{c}{d}$; therefore (Art. 67),

$$\frac{ma}{mb} = \frac{\frac{c}{n}}{\frac{d}{n}}; \qquad \text{whence, } ma : mb :: \frac{c}{n} : \frac{d}{n}.$$

175. VII. If the first and third terms of a proportion be multiplied or divided by any quantity, as also the second and fourth, the quantities which result will be proportional.

For $$\frac{ma}{b} = \frac{mc}{d}, \text{ and } \frac{\frac{ma}{b}}{n} = \frac{\frac{mc}{d}}{n};$$

whence, $$ma : \frac{b}{n} :: mc : \frac{d}{n}.$$

176. VIII. In any number of proportions, if the corresponding terms be multiplied together, the products will be proportional.

Let there be any number of proportions,

$$\frac{a}{b} = \frac{c}{d}, \frac{e}{f} = \frac{g}{h}, \frac{i}{k} = \frac{l}{m};$$

then

$$\frac{aei}{bfk} = \frac{cgl}{dhm};$$

hence, $$aei : bfk :: cgl : dhm.$$

177. PROB. I. *To find the sum of an equidifferent series.* Multiply the sum of the extremes by the number of terms, and take half the product.

Let a be the first term, d the common difference, and s the sum of the series; then

$s = a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d)+(a+6d)$, &c.

Now, the last or nth term is $a+(n-1)d$, the last but one is $a+(n-2)d$, the last but two $a+(n-3)d$; also, the first $a+(n-n)d$. Hence, by reversing the terms, the series may be thus written:

$s = [a+(n-1)d]+[a+(n-2)d]+[a+(n-3)d]$ ..... $+[a+(n-n)d]$.

Adding this series to the last, we find

$2s = [2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d]+$ &c. to n terms.

Whence, as each term is $2a+(n-1)d$, and as there are n terms, we have

$$2s = n[2a+(n-1)d],$$

and therefore $$s = \frac{n}{2}[2a+(n-1)d]. \quad (2)$$

Now let the last term

$$a+(n-1)d = l, \quad (1)$$

therefore $$s = \frac{n}{2}(a+l). \quad (3)$$

178. *Cor.* Hence, and, from Art. 165, it follows that, if m be the middle term of a series, $s = nm$, (1). Also, if p and q be terms equally distant from the extremes, $s = \frac{n}{2}(p+q)$, (2). These two formulæ facilitate, in many cases, the summation of series.

179. *Cor.* From the equations $s = \frac{n}{2}(a+l)$, and $l = a+(n-1)d$, any two of the quantities, s, n, a, l, d, may be found, when the other three are given.

EXAMPLES.

1. Find the sum of 12 terms of the series

3, 7, 11, 15, &c.

Here, $a = 3$, $d = 4$, $n = 12$, $l = 3+(12-1)4 = 47$;
hence, $s = \frac{12}{2}(3+47) = 50 \times 6 = 300$.

2. If the first term of a progression be 14, and 28 the sum of 8 terms, what is the common difference?

Here, from the formula,

$$28 = \frac{8}{2}(28+7d),$$

we find $d = -3$;
whence, the progression is

14, 11, 8, 5, 2, —1, —4, —7, &c.

3. If the sum of an increasing series be 147, the first term 7, and common difference 7; required the number of terms.

Substituting these numbers in the formula for s, we find, after some reductions, $n^2+n = 42$; a quadratic, which gives $n = 6$, the number of terms required.

180. II. *To find the sum of a series of continual proportionals.* Multiply the last term by the ratio, and divide the difference between this product and the first term by the difference between the ratio and unity; observing that the last term becomes zero, in an infinite decreasing series.

Let a be the first term, r the ratio, n the number of terms, and s the sum of the series; then

$$s = a+ar+ar^2+ar^3+ar^4 \ldots . +ar^{n-1}.$$

Multiplying both sides by r, and from the result subtracting this equation, we have

$$sr - s = ar^n - a;$$

whence, $$s = \frac{ar^n - a}{r - 1}: \quad (1)$$

putting $ar^{n-1} = l$, this becomes

$$s = \frac{rl - a}{r - 1}. \quad (2)$$

When r is a fraction, as n increases, r^n and ar^n will decrease; and when n becomes infinite, the last term will become less, with respect to a, than any thing assignable, and is therefore to be taken as nothing: in this case therefore, we have

$$s = \frac{-a}{r - 1} = \frac{a}{1 - r}. \quad (3)$$

181. *Cor.* From the equation $s = \frac{rl - a}{r - 1}$, any three of the quantities s, r, l, a, being given, the fourth may be found. See "*Application of Logarithms.*"

EXAMPLES.

1. Find the sum of 8 terms of the series

2, 6, 18, 54, &c.

Here $a = 2$, $r = 3$, $n = 8$;

we have, therefore, by substitution in formula (1),

$$s = \frac{ar^n - a}{r - 1} = \frac{2 \times 3^8 - 2}{3 - 1} = 6560.$$

2. Find the sum of the infinite series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8}, \text{ &c.}$$

Here $a = 1$, $r = -\frac{1}{2}$,

hence, from formula (3),

$$s = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$ *See* Art. 81, 82.

3. Insert three means between the extremes 4 and 324 Let the terms be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4;$$

then $a = 4$, and $ar^4 = 324$; therefore $r^4 = 81$, and hence $r = 3$: the intermediate terms, therefore, are 12, 36, 108.

182. III. *To find an harmonical mean, and an harmonical third proportional.*

Let a, b, c, be three harmonical proportionals; then (Art. 163),

$$a : c :: a-b : b-c;$$

and therefore (Art 167),

$$ab+bc = 2ac, \text{ and } b = \frac{2ac}{a+c}: \quad (1)$$

that is, an harmonical mean between two quantities is twice their product divided by their sum.

From the same equation we deduce,

$$c = \frac{ab}{2a-b}, \quad (2)$$

which is a third harmonical proportional.

183. THEOR. The reciprocals of quantities in harmonical progression are in arithmetical progression.

Let a, b, c, d, &c., be in harmonical progression, then $a : c :: a-b : b-c$; therefore (Art. 167) $ab-ac=ac-bc$, Dividing both members by abc, we find

$$\frac{1}{c}-\frac{1}{b} = \frac{1}{b}-\frac{1}{a}$$

Again $b : d :: b-c : c-d$; therefore $bc-bd=db-dc$. Dividing by bcd, we have $\frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}$. now $\frac{1}{c}-\frac{1}{b}$ has been shown equal to $\frac{1}{b}-\frac{1}{a}$. Thus the quantities $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, which are the reciprocals of a, b, c, and d, have a common difference, and are therefore in arithmetical progression.

184. *Cor.* 1. Since the sum of any number of quantities, is not deducible from the sum of their reciprocals, there can be no general expression for the sum of an harmonic series.

185. *Cor.* 2. Hence we can find the nth term of an harmonic series. For if a and b be the first two terms of such a series, $\frac{1}{a}$ and $\frac{1}{b}$ are the contiguous terms of an arithmetic progression, and therefore $\frac{1}{b}-\frac{1}{a}$ is the common difference; hence (Art. 177, 1,) the nth term is $\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{(n-1)a-(n-2)b}{ab}$.

The nth term of the harmonic series therefore is

$$\frac{ab}{(n-1)a-(n-2)b}.$$

186. *Cor.* 3. Hence also if the extremes, a and l, and the number of terms n be given, the intervening terms can be found; for since the reciprocals are in arithmetic progression, $\frac{1}{l}$ will be the last term, and b being the

common difference, we shall have

$$\frac{1}{l} = \frac{1}{a} + (n-1)b.$$

hence $$b = \frac{a-l}{(n-1)al}.$$

The arithmetic means are therefore

$$\frac{1}{a} + \frac{a-l}{(n-1)al}, \quad \frac{1}{a} + \frac{2(a-l)}{(n-1)al}, \quad \&c.$$

adding these fractions, and then taking the reciprocals, we obtain for the harmonic means,

$$\frac{(n-1)al}{a+(n-2)l}, \quad \frac{(n-1)al}{2a+(n-3)l}, \quad \frac{(n-1)al}{3a+(n-4)l} \quad \&c.$$

SUMMATION OF SERIES BY THE METHOD OF INDETERMINATE COEFFICIENTS.

187. The following simple method of summing series, will be found of very general application. It presupposes a knowledge of the nth term; but this can be readily ascertained in most cases by inspection, or by the rule for finding the nth term of an arithmetic series. It is always applicable when the nth term can be expressed by a series of positive and integral powers of n.

EXAMPLES.

1. Required the sum of the series $1^3+3^3+5^3$ &c.....to n terms.

Since (Art. 177,) the nth term of the series

$$1+3+5 \text{ \&c. is } 2n-1,$$

the nth term of this series will be $(2n-1)^3$.

Assume

$$1^3+3^3+5^3+\&c.\ldots.(2n-1)^3=An^4+Bn^3+Cn^2+Du.*(1)$$

Now since the series is true for all values of n, it will be true when n becomes $n+1$; substituting $n+1$ for n, we have, since the $(n+1)$th term of the series

$$1+3+5 \text{ \&c. is } 2n+1,$$

$$1^3+3^3+5^3+\&c.\ldots.(2n-1)^3+(2n+1)^3=A(n+1)^4+B(n+1)^3+C(n+1)^2+D(n+1) \quad (2.)$$

Subtracting series (1) from (2), developing the powers and incorporating, we find

$$8n^3+12n^2+6n+1=4An^3+(6A+3B)n^2+(4A+3B+2C)n+(A+B+C+D).$$

By equating the coefficients we deduce the following values,

$$A=2,\ B=0,\ C=-1,\ D=0,$$

which being substituted in (1) give

$$1^3+3^3+5^3+\&c.\ldots.(2n-1)^3=2n^4-n^2=n^2(2n^2-1)$$

for the required sum.

2. Required the sum of the series

$$1\times3+3\times5+5\times7\ldots.\text{to } n \text{ terms.}$$

The first factor of each of these being the terms of the series 1, 3, 5, &c., the first factor of the last term will be as before $(2n-1)$: the second factor evidently exceeds the first by 2; the last term therefore is $(2n-1)(2n+1)$.

* The reason of this assumption is manifest; there can be no term which does not contain n: for, if there were, it would remain when $n=0$, but this is impossible, as the series then becomes 0. Nor can there be a higher power than n^4; because the highest power in the greatest or nth term is n^3, therefore, n being the number of terms, the sum of the series must be less than $n(2n-1)^3$, the highest power in which is n^4.

Assume

$$1\times3+3\times5+5\times7+\text{ \&c.}\ldots\ldots(2n-1)(2n+1)=An^3+Bn^2+Cn$$

substituting $n+1$ for n we have,

$$1\times3+3\times5+5\times7+\text{ \&c.}\ldots..(2n-1)(2n+1)+(2n+1)(2n+3)=A(n+1)^3+B(n+1)^2+C(n+1)$$

subtracting, developing, and equating the coefficients we find

$$A=\tfrac{4}{3},\ B=2,\ C=-\tfrac{1}{3}$$

which, being substituted, give for the required sum

$$\frac{4n^3}{3}+2n^2-\frac{n}{3}=\frac{n}{3}(4n^2+6n-1).$$

Similarly all series of this kind may be summed *by first finding the* nth *term, and assuming for the result, a series beginning with a power of* n, *whose index exceeds by unity, the highest index in the* nth *term.*

SUMMATION OF SERIES BY SUBTRACTION.

188. Certain arithmetic series may be readily summed by the following method, which will be best understood from a few examples.

EXAMPLES.

1. Required the sum of the series

$$\frac{1}{2.5.8}+\frac{1}{5.8.11}+\frac{1}{8.11.14.}+\text{ \&c. to infinity.}$$

Omitting the last factor from the denominator of each term assume

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}\ \text{\&c.}=m.$$

hence, by transposition

$$\frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14} + \&c. = m - \frac{1}{2.5}$$

subtracting these equations from each other, term by term, we have

$$\frac{6}{2.5.8} + \frac{6}{5.8.11} + \frac{6}{8.11.14} + \&c. = \frac{1}{10};$$

therefore

$$\frac{1}{2.5.8} + \frac{1}{5.8.11} + \frac{1}{8.11.14} + \&c. = \frac{1}{60}$$

To find the sum of n terms is somewhat more difficult, the method is as follows:

As the nth term is evidently equal to

$$\frac{1}{(3n-1)(3n+2)(3n+5)},$$

we commence by assuming

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \&c. \ldots\ldots\ldots \frac{1}{(3n-1)(3n+2)} + \frac{1}{(3n+2)(3n+5)} = m \therefore \frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14} + \&c. \ldots\ldots\ldots\ldots \frac{1}{(3n+2)(3n+5)} = m - \frac{1}{2.5}$$

Proceeding as before we find

$$\frac{6}{2.5.8} + \frac{6}{5.8.11} + \frac{6}{8.11.14} + \&c. \ldots\ldots\ldots\ldots \frac{6}{(3n-1)(3n+2)(3n+5)} + \frac{1}{(3n+2)(3n+5)} = \frac{1}{10}; \therefore \frac{6}{2.5.8} + \frac{6}{5.8.11} + \&c. \ldots\ldots \frac{6}{(3n-1)(3n+2)(3n+5)} = \frac{1}{10} - \frac{1}{(3n+2)(3n+5)}; \therefore \frac{1}{2.5.8} + \frac{1}{5.8.11} + \&c. \ldots\ldots \frac{1}{(3n-1)(3n+2)(3n+5)} = \frac{1}{60} - \frac{1}{6(3n+2)(3n+5)}.$$

If in this result we make n infinite we find, as before, the sum of the series to infinity to be $\frac{1}{60}$.

It is necessary in finding the sum of n terms to continue the first series to $n + 1$ terms, in order that it may coincide with the second series, except in the first term; and the second series, being formed by rejecting the first factor, from the denominator of each term, must end in

$$\frac{1}{(3n+2)(3n+5)}.$$

2. Sum the series

$$\frac{2}{1.3.5.7} + \frac{3}{3.5.7.9} + \frac{4}{5.7.9.11} + \text{\&c. to infinity.}$$

As before assume

$$\frac{2}{1.3.5} + \frac{3}{3.5.7} + \frac{4}{5.7.9} + \text{\&c. to infinity} = m$$

$$\therefore \frac{2}{3.5.7} + \frac{3}{5.7.9} + \frac{4}{7.9.11} + \text{\&c. to infinity} = m - \frac{2}{1.3.5} - \left(\frac{1}{3.5.7} + \frac{1}{5.7.9} + \frac{1}{7.9.11} + \text{\&c. to infinity.}\right)$$

The second equation is obtained by observing that the numerator of each term is less by 1 than the numerator of that term in the upper series, which has the same denominator; consequently we have to subtract the series included in the vinculum from both sides. Subtracting as before, the lower series from the upper we have,

$$\frac{12}{1.3.5.7} + \frac{18}{3.5.7.9} + \frac{24}{5.7.9.11} + \text{\&c.} = \frac{2}{15} + \left(\frac{1}{3.5.7} + \frac{1}{5.7.9} + \frac{1}{7.9.11} + \text{\&c. to infinity.}\right)$$

This latter series may be found, as in the last example,

$$= \frac{1}{60}.$$

$$\therefore \frac{12}{1.3.5.7} + \frac{18}{3.5.7.9} + \frac{24}{5.7.9.11} = \frac{2}{15} + \frac{1}{60} = \frac{3}{20},$$

$\therefore$ dividing both sides by 6, we find

$$\frac{2}{1.3.5.7} + \frac{3}{3.5.7.9} + \frac{4}{5.7.9.11} + \&c. = \frac{1}{40}.$$

The sum of n terms may be found as before.

The rule may be briefly stated as follows: *assume a series* = m, *omitting the last factor from the denominator of each term; from it subtract another series, formed by omitting the first factor from the denominator of each term of the original series; and then proceed as in the above examples.*

PILING OF BALLS.

189. As an additional illustration of the above theory, the method of finding the number of balls in a pile may be adduced. These piles are either *square, triangular,* or *rectangular;* the balls being all of equal size.

The square pile is a square pyramid, and each layer of balls is a square whose side exceeds that of the one above it by one ball. Hence, the number of layers is the same as the number of balls in one side of the lowest layer: on the top there is evidently only one ball. It is therefore plain that the number of balls will be the sum of n terms of the series $1^2+2^2+3^2+$ &c. .... n^2, n being the number of layers. This sum, found by the preceding method, will be $\frac{n(n+1)(2n+1)}{2.3}$.

190. In the triangular pile the base, and consequently each layer, is an equilateral triangle, whose side exceeds that of the one above it by one ball, there being plainly

only one ball on the top. Each of the horizontal layers consists of a number of rows, regularly decreasing by unity from the number in the outside row down to 1. Hence it is plain that the number in the nth layer will be the sum of the series,

$$\ldots\ldots\ldots\ldots\ 1+2+3+\ldots\ldots\ldots n=\frac{n(n+1)}{2}.$$

$$\ldots\ldots\ldots\ (n-1)\text{th}\ \ldots\ 1+2+3+\ldots(n-1)=\frac{(n-1)n}{2}$$

$$\ldots\ldots\ (n-2)\text{th}\ldots\ 1+2+3+\ \ldots(n-2)=\frac{(n-2)(n-1)}{2}$$

&c. &c. &c.

In the second layer the number will be $1+2\ \ldots\ =\frac{2\times 3}{2}$

.... first ..... 1 $\ldots=\frac{1\times 2}{2}$.

Hence the series we have to sum is

$$\frac{1\times 2+2\times 3+\ \ldots\ldots\ (n-1)n+n(n+1)}{2};$$

where n denotes the number of layers, or, which is the same thing, the number of balls in one side of the bottom layer. The sum of this series will be readily found to be

$$\frac{n(n+1)(n+2)}{2.3}.$$

191. In the rectangular pile, each layer is a rectangle, whose sides respectively exceed the sides of the one above it by one ball, the number of layers being evidently the same as the number of balls in the shorter side of the base. Let $n=$ this number, and $n+m=$ the number on the longer side, where m denotes the difference, which manifestly remains the same in all the layers. The top layer is a single row of balls, but for the

sake of uniformity it may be considered a rectangle, whose sides are 1 and $1+m$. The number of balls in the bottom layer is $n(n+m)$; in the next, $(n-1)(n+m-1)$, and so on to the top, which is $1(1+m)$: hence the series we have to sum is

$$1(1+m)+2(2+m)+3(3+m)+\ldots\ldots\ n(n+m).$$

The simplest method of doing this is the following: taking off the brackets it becomes

$$1^2+2^2+3^2+\ \&c.\ldots\ldots\ldots\ +n^2$$
$$+m(1+2+3+\ \&c.\ldots\ldots\ldots\ n)$$

The former series has been already shown

$$=\frac{n(n+1)\ (2\,n+1)}{6}$$

and the latter $=\frac{n(n+1)}{2}$; hence the sum is

$$\frac{n(n+1)(2n+1)}{6}+m\frac{n(n+1)}{2}=\frac{n(n+1)}{6}\ (2n+1+3m);$$

or if instead of $n+m$, the longest side of the base, we write l, we have

$$\frac{n(n+1)(3l-n+1)}{6}.$$

If $l=n$, the pile becomes square, and we have the same result as before,

$$\frac{n(n+1)(2n+1)}{6}.$$

We might have obtained the above result in another manner; for if we cut off n rows from the longer side of the base, by a line terminating at the first ball on the top, we shall have a square pile of n layers whose sum is

$$\frac{n(n+1)(2n+1)}{6}$$

and m triangles reclining on it; each consisting of n rows of balls, whereof n is the number in the longest, and 1 in

the shortest side; or in other words, each triangle is the sum of the series,

$$1+2+3+\ \&c.\ldots n=\frac{n(n+1)}{2},$$

and as there are m of these triangles, the sum of them all is $m\,\frac{n(n+1)}{2}$, and therefore the number of balls in the whole pile is, as before, equal to

$$\frac{n(n+1)(2n+1)}{6}+m\frac{n(n+1)}{2}=\frac{n(n+1)(3l-n+1)}{6}.$$

192. Should the pile be incomplete, the easiest method of finding the number of balls is, first, to find the number in the complete pile, and afterwards to find the number in the wanting pile, and to subtract the one sum from the other. Thus, if it were proposed to find the number of balls in an incomplete triangular pile of 21 layers, one side of the bottom layer being 40; we first find the number there would be in the pile, if complete, which is found by the formula,

$$\frac{n(n+1)(n+2)}{6}=\text{(in this case)}$$

$$\frac{40\times41\times42}{6}=40\times41\times7=11480:$$

the number of layers ought to be 40, but by hypothesis there are only 21, therefore there are 19 wanting, consequently the number of balls wanting to complete the pile

$$\text{is}\ \frac{19\times20\times21}{6}=19\times10\times7=1330.$$

Hence the number of balls in the incomplete pile is 11480 $-1330=10150$.

QUESTIONS IN PROGRESSION.

1. Four numbers are in geometrical progression; the sum of the first and second is 15, and the sum of the third and fourth is 60. Required the numbers.

Let x, xy, xy^2, xy^3, be the numbers; then

$$x+xy=15,$$

$$xy^2+xy^3=60,$$

or $$y^2(x+xy)=60;$$

that is, $$15y^2=60:$$

hence, $$y=\pm 2,$$

and therefore $$x=5.$$

The numbers, therefore, are

$$5,\ 10,\ 20,\ 40.$$

2. The base of a right-angled triangle is 6, and the sides are in arithmetical progression. Required the sides.

Let the sides be $6-x$, 6, $6+x$; then (Eucl. I. 47),

$$36-12x+x^2+36=36+12x+x^2.$$

Hence we find $x=\frac{5}{2}$; and the sides therefore are

$$4\tfrac{1}{2},\ 6,\ 7\tfrac{1}{2}.$$

But if the first term of the progression be 6, and the other sides $6+x$, $6+2x$, then

$$36+24x+4x^2=36+36+12x+x^2;$$

hence, $$x^2+4x=12.$$

Solving this equation, we find

$$x=2,\text{ or }-6;$$

and the sides, therefore, are

$$6,\ 8,\ 10.$$

The ambiguity here arises from the data of the problem

not being sufficiently limited: the algebraic language is more precise than that of the problem.

3. Two persons, A and B, set out together from London, to travel round the world, a distance of 23661 miles, the one going east and the other west. A goes one mile the first day, two the second, and so on. B goes 20 miles a day. In how many days will they meet, and how far will each travel?

Let x be the number of days; then B goes $20x$ miles: and to find the number A goes, we must sum the series whose first term is 1, common difference 1, and number of terms x; this is

$$\frac{x}{2}(1+x) = \text{number of miles A goes.}$$

Hence, by the question,

$$\frac{x^2+x}{2} + 20x = 23661.$$

Solving this quadratic, we find

$$x = 198, \text{ or } -239.$$

They travel 198 days; A goes 19701, and B 3960 miles.

4. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water, the quantities of which form an equidifferent series, and the number of gallons of brandy and rum together is to the number of gallons of rum and water together as 8 to 9. Required the quantity of each.

Let $x-y$, x, $x+y$, denote the quantities of brandy, rum, and water; then, by the question,

$$3x = 51, \text{ and } \therefore x = 17,$$

and $$2x-y : 2x+y :: 8 : 9;$$

or substituting for x its value

$$34-y : 34+y :: 8 : 9$$

whence, $y = 2$, and therefore $x = 15$. The quantities, therefore, are 15, 17, 19.

EXERCISES.

1. Find the sum of n terms of the series of natural numbers, 1, 2, 3, 4, 5, &c. *Ans.* $\frac{n(n+1)}{2}$.

2. Find the sum of n terms of the series of odd numbers, 1, 3, 5, 7, &c. *Ans.* n^2.

3. Find the sum of n terms of the series of even numbers, 2, 4, 6, 8, &c. *Ans.* $n(n+1)$.

4. Required the sum of 9 terms of the progression 11, 9, 7, &c. *Ans.* 27.

5. If the sum of 9 terms of an equidifferent series be 171, and the common difference 4, what is the first term? *Ans.* 3.

6. If the first term of a progression be 2, and the sum of 8 terms 100, what is the common difference? *Ans.* 3.

7. Insert three arithmetical means between the extremes $\frac{1}{3}$ and $\frac{1}{2}$. *Ans.* $\frac{5}{8}$, $\frac{6}{12}$, $\frac{11}{24}$.

8. Insert two arithmetical means between —3 and 3. *Ans.* —1 and 1.

9. Find the sum of 7 terms of the series, 1, 2, 4, 8, &c. *Ans.* 127.

10. Find the sum of 12 terms of the series, 1, —3, 9, —27, &c. *Ans.* —132860.

11. Find the sum of the infinite series, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. *Ans.* 2.

12. Find the sum of 20 terms of the series, 1, 2, 4, 8, &c. *Ans.* $2^{20} - 1$.

13. If the first term of an arithmetical progression be $3\frac{1}{3}$, the common difference $1\frac{4}{5}$, and the sum 22, find the number of terms. *Ans.* 4.

14. Find the sum of $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+\frac{1}{15}$, &c. to 7 terms; and of $1+\frac{5}{2}+2+\frac{5}{2}$, &c. to 6 terms. *See Art.* 178.

Ans. $\frac{7}{15}$ and $13\frac{1}{2}$.

15. Sum the series $1+3+9$, &c. to 9 terms.

Ans. 9841.

16. Sum the two series $1-\frac{2}{5}+\frac{4}{25}$, &c. and $4+3+\frac{9}{4}$, &c. to infinity. *Ans.* $\frac{5}{7}$ and 16.

17. Sum each of the series, $3+\frac{9}{2}+6\ldots$. and $3+7+11$ .... to 20 terms. *Ans.* 345 and 820.

18. Find the 12th term and the sum of 12 terms of the series, $1+3+9+27+\ldots$. *Ans.* 177147 and 265720.

19. How many terms of the series, 19, 17, 15, &c., amount to 91? *Ans.* 7, and 13.

20. Insert three geometric means between 39 and 3159.

Ans. 117, 351, 1053.

21. Find the sum of n terms of each of the series $13+12\frac{2}{3}+12\frac{1}{3}+\ldots$., and $\frac{1}{2}-\frac{2}{3}-\frac{1}{6}-\ldots$..

Ans. $\frac{n}{6}(79-n)$, and $\frac{n}{12}(13-7n)$.

22. In an arithmetical progression the 5th and 9th terms are 13 and 25, find the 7th term. *Ans.* 19.

23. How many terms of the series .034, .0344, .0348, amount to 2.748. *Ans.* 60.

24. Sum the series $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ldots$. to n terms.

Ans. $\frac{n-1}{2}$.

25. Insert three arithmetic means between 117 and 477.

Ans. 207, 297, and 387.

26. There are n arithmetic means between 1 and 31, such that the 7th : $(n-1)$th : : 5 : 9; required n, and write out the series.

Ans. $n = 14$; and the series is 1, 3, 5, &c.

27. Insert three geometric means between the extremes 5 and 405. *Ans.* 15, 45, 135.

28. Find the sum of the infinite series, $\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4}$, &c. *Ans.* $\frac{1}{x+1}$.

29. Find the sum of the infinite series, $\frac{x}{a} - \frac{bx}{a^2} + \frac{b^2x}{a^3} - \frac{b^3x}{a^4}$, &c. *Ans.* $\frac{x}{a+b}$. *See* Ex. 1, *Art.* 81.

30. Required a geometric mean between 20 and 45.

Ans. 30.

31. Find an harmonical mean between 2 and 6.

Ans. 3.

32. Required an harmonic third proportional to 6 and 8.

Ans. 12.

33. Find the value of the decimal .333 &c.

Ans. $\frac{1}{3}$.

34. Sum the mixed circulating decimal ·247′95′.

Ans. $\frac{8257}{33300}$.

35. If P and Q denote the pth, and qth terms respectively of an arithmetic series, find the first term, and the sum of series.

Ans. $a = \frac{Q(p-1) - P(q-1)}{p-q}$,

$$s = \frac{Q(2p-n-1) - P(2q-n-1)}{p-q} \cdot \frac{n}{2}.$$

36. If the first and second terms of an harmonic series be 5 and 4, find the tenth term. *Ans.* $\frac{20}{15}$.

37. Insert two harmonic means between 6 and 24.

Ans. 8 and 12.

38. The sum of three consecutive terms in harmonic progression is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series.

Ans. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

39. In a geometric series the $(p+q)$th term $= m$, and the $(p-q)$th $= n$; find the pth term and also the qth.

Ans. $p\text{th} = \sqrt{mn}$, $q\text{th} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

40. In a geometric progression if P and Q denote the pth and qth terms, find the nth.

Ans. $\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

41. The difference between two numbers is 48, and the arithmetic mean exceeds the geometric by 18; find the numbers. *Ans.* 49 and 1.

42. If the $(p+q)$th term of an arithmetic series $= m$, and the $(p-q)$th $= n$, find the pth term and also the qth.

Ans. $\frac{m+n}{2}$, and $m - \frac{p}{2q}(m-n)$.

43. Sum the series, $\frac{1}{1.3.5.7} + \frac{1}{3.5.7.9} + \frac{1}{5.7.9.11}$ to infinity, and also to n terms.

Ans. $\frac{1}{90}$, and $\frac{n(4n^2+18n+23)}{3.1.3.5(2n+1)(2n+3)(2n+5)}$.

44. What is the common fraction answering to the decimal .2424 &c. *Ans.* $\frac{8}{33}$.

45. The sum of three continual proportionals is 35, and the mean is to the difference of the extremes as 2 to 3. Required the numbers. *Ans.* 5, 10, 20.

46. A gentleman divided £210 among three servants, the shares being in geometric progression; and the first had £90 more than the last. How much had each?

Ans. £120, £60, £30.

47. Of three continual proportionals the sum is 14, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Ans. 2, 4, 8.

48. From two towns 168 miles distant, two persons, A and B, set out to meet: A went 3 miles the first day, 5 the second, 7 the third, and so on: B went 4 miles the first day, 6 the second, 8 the third, and so on. In how many days did they meet? *Ans.* 8.

49. Four numbers form an equidifferent series; their sum is 32, and the sum of their squares is 276. Required the numbers. *Ans.* 11, 9, 7, 5.

50. A traveller sets out and travels 1 mile the first day, 2 the second, 3 the third, and so on: in 5 days after, another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Ans. 3 days, and 36 miles.

51. A person bought 47 sheep, and gave 1 shilling for the first, 3 for the second, 5 for the third, and so on. What did all the sheep cost? *Ans.* £110 9s.

52. If a body move for ever as follows: namely, 20 miles the first minute, 19 miles the second minute, $18\frac{1}{20}$, the third, and so on in geometric progression. Required the greatest distance it can reach. *Ans.* 400 miles.

53. A number consists of three digits, which are in arithmetica l progression; and thenumber divided by the

sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. Required the number.

Ans. 234.

54. The sum of three numbers in harmonical proportion is 13, and the product of their extremes is 18. Required the numbers. *Ans.* 6, 4, 3.

55. Sum the series $1^3+2^3+3^3$ &c. to n terms.

Ans. $\left(\frac{n(n+1)}{2}\right)^2$.

56. Sum the series $1\times2+2\times3+3\times4+$ &c.

Ans. $\frac{n(n+1)(n+2)}{3}$.

57. Show that $1^3+2^3+3^3+\ldots n^3=(1+2+3+\ldots n)^2$

58. Sum the series $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}$ &c. to infinity.

Ans. $\frac{1}{6}$.

59. Sum the series $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}$ &c. to n terms, and also to infinity.

Ans. $\frac{n}{3(2n+3)}$, and $\frac{1}{6}$.

60. Find the number of balls in a triangular pile of 30 layers; and also in a square pile.

Ans. 4960, and 9455.

61. Find the same in the rectangular pile, length of base = 60, and breadth = 30. *Ans.* 23405.

62. Sum the series $1^2+3^2+5^2+\ldots\ldots(2n-1)^2$.

Ans. $\frac{(2n-1)2n(2n+1)}{1.2.3}$.

63. Sum the series $1^2\times4^3+3^2\times6^3+5^2\times8^3$ &c.

Ans. $\frac{80n^6+432n^5+710n^4+180n^3-370n^2-72n}{15}$.

64. Sum the series $1\times3^2+3\times5^2+5\times7^2+\ldots\ldots(2n-1)(2n+1)^2$.

Ans. $n^2(2n^2+3)+\frac{n}{3}(16n^2-4)$.

65. Find the number of balls in a rectangular pile, the length and breadth of the lowest row being 52 and 34.

Ans. 24395.

66. Find the number of balls in an incomplete square pile, the side of the base being 24, and of the top 8.

Ans. 4760.

67. Find the same in a complete triangular pile of 20 layers, and an incomplete rectangular pile of 12 layers, the length and breadth of the base being 40 and 20.

Ans. 1540, and 6146.

68. Show that the number of balls in the two lowest layers of a complete triangular pile, equals the number in the base of a complete square pile of the same number of layers.

69. There are two complete piles of balls, one triangular and the other square, such that the whole number of balls in the former is to the whole number in the latter, as 6 is to 11. Required the number of balls in each pile, the bottom row of both having the same number of balls in a side.

Ans. 816 in the triangular; 1496 in the square pile.

70. Sum the series $\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ &c. ad infinitum.

Ans. $\frac{1}{12}$.

71. Sum the series $\frac{1}{2.5.8}+\frac{3}{5.8.11}+\frac{5}{8.11.14}$ +&c. to infinity.

Ans. $\frac{7}{180}$.

72. Find the sum of the series $1+2r+3r^2+4r^3+$ &c. to infinity.

Ans. $\frac{nr^{n+1}-(n+1)r^n+1}{(1-r)^2}$.

73. Find the sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4\,5.6}+$ &c. to infinity.

Ans. $\frac{3}{2}$.

74. Sum the series $\frac{1}{1.3.5.7}+\frac{2}{3.5.7.9}+\frac{3}{5.7.9.11}+$ &c. ad infinitum.

Ans. $\frac{1}{72}$.

75. Sum the series $1^5+2^5+3^5+\ldots$ to n terms.

Ans. $\frac{2n^6+6n^5+5n^4-n^2}{12}$.

76. Find the sum of n terms of the series $\frac{n-1}{n}x+\frac{n-2}{n}x^2+\frac{n-3}{n}x^3+$ &c.

Ans. $\frac{(n-1)x-nx^2+x^{n+1}}{n(1-x)^2}$.

77. Sum the series $\frac{1}{\sqrt{2}(1+\sqrt{2})}+\frac{1}{(1+\sqrt{2})(2+\sqrt{2})}+\frac{1}{(2+\sqrt{2})(3+\sqrt{2})}+$ &c. to infinity.

Ans. $\frac{1}{\sqrt{2}}$.

78. If s_1, s_2, s_3,..... s_r, be the sums of r arithmetic series, each continued to n terms, the first terms being 1, 2, 3,.... and the common differences 1, 3, 5,.... respectively, required the sum of s_1, s_2, s_3.... s_r.

Ans. $\frac{nr}{2}(nr+1)$.

79. Sum the series $1.2.3+2.3.4+3.4.5+$ &c. to n terms.

Ans. $\frac{n(n+1)(n+2)(n+3)}{4}$.

CHAP. XII.—LOGARITHMS.

ART. 193. DEF. I. When the several terms of a series of numbers are considered as powers of some common root, the indices of those powers are called the *Logarithms* of the numbers to which the powers are equal. Thus, if we consider $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, &c. as powers of the common root 2, we shall have

$$\begin{aligned} \tfrac{1}{16} &= 2^{-4}, & 2 &= 2^{1}, \\ \tfrac{1}{8} &= 2^{-3}, & 4 &= 2^{2}, \\ \tfrac{1}{4} &= 2^{-2}, & 8 &= 2^{3}, \\ \tfrac{1}{2} &= 2^{-1}, & 16 &= 2^{4}, \\ 1 &= 2^{0}, & 32 &= 2^{5}, \end{aligned}$$

&c.

where the indices —4,—3,—2,—1, 0, 1, 2, 3, &c. are the logarithms of the numbers $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, &c.

But the same numbers may be considered as powers of the common root 4, and we shall then have

$$\begin{aligned} \tfrac{1}{16} &= 4^{-2}, & 4 &= 4^{1}, \\ \tfrac{1}{8} &= 4^{-\frac{3}{2}}, & 8 &= 4^{\frac{3}{2}}, \\ \tfrac{1}{4} &= 4^{-1} & 16 &= 4^{2}, \\ \tfrac{1}{2} &= 4^{-\frac{1}{2}}, & 32 &= 4^{\frac{5}{2}}, \\ 1 &= 4^{0}, & 64 &= 4^{3}, \\ 2 &= 4^{\frac{1}{2}}; \end{aligned}$$

&c.

where the indices —2,—$\frac{3}{2}$,—1, &c. are the logarithms of the corresponding numbers.

194. *Schol.* If it be required to find what power of 4 is equal to 13, it will appear from the table that since 13 is between 8 and 16, the index of 4, that is, the logar-

ithm of 13, will be between $\frac{3}{2}$ and 2; and, from Arts. 86, 100, that if such an index cannot be exactly found, it may be approximated to. Hence, logarithms may be calculated for all numbers.

195. II. The common root, of which other numbers are taken as powers, is called the *base* of the system of logarithms. Thus, in the examples of Art. 193, the bases are 2 and 4.

196. *Schol.* Any number may be assumed as the base; and it is plain that every base will give a different system of logarithms: but the base which has been found most convenient is 10. Logarithms constructed to the base 10 are called, for the sake of distinction, *common logarithms.* Hence, *the common logarithm of a number is the index of the power of* 10 *which is equal to that number.* The only other base to which logarithms have ever been constructed, is the number 2·718281828459... This was the base chosen by Baron Napier, the inventor of logarithms; and hence these are called Napierian logarithms. Since $a^0=1$ it is plain that, in every system, the logarithm of 1 is 0.

Taking, therefore, 10 for the base, its powers will be as follows:

$10^0=1$, and $10^{-1}=\frac{1}{10}=\cdot1$;
$10^1=10$, $10^{-2}=\frac{1}{100}=\cdot01$;
$10^2=100$, $10^{-3}=\frac{1}{1000}=\cdot001$;
$10^3=1000$, $10^{-4}=\frac{1}{10000}=\cdot0001$;
$10^4=10000$, $10^{-5}=\frac{1}{100000}=\cdot00001$;
&c.

The several indices are here the common logarithms of

the corresponding powers.* It is plain that the logarithms of all numbers between 1 and 10, that is, numbers of one figure, are between 0 and 1; that the logarithms of numbers between 10 and 100, that is, those consisting of two figures, are between 1 and 2; and in general that the integral part of the logarithm, which is usually called the *characteristic*, is always a unit less than the number of digits. Also, it is plain that of decimals between ·1 and ·01, the logarithms are greater than —2, and less than —1, or that they are —2 increased by some fraction; and those of decimals between ·01 and ·001, are —3 increased by some fraction; and, in general, that the logarithm of a decimal consists of a positive fraction combined with a negative integer, greater by 1 than the number of zeros prefixed to the significant figures of the decimal. Thus if —3+·726401 be a logarithm, the corresponding number will be a decimal fraction, having two zeros prefixed to its significant figures. It is usual to write such logarithms with the sign — over the integral part, omitting the sign + before the fractional part. Thus $\overline{3}$·726401 means —3+·726401.†

* By reasoning, as in Art. 194, it might be shown from the above table, that logarithms may be approximated to when the numbers are not exact powers of 10. This will also appear from the following example. Suppose the logarithm of 5 to be required, or from $10^x = 5$ to find x; first, x is plainly greater than $\frac{1}{2}$; let $x = \frac{3}{5}$, then since $10^{\frac{3}{5}} = \sqrt[5]{100}$, and $5 = \sqrt[3]{125}$, $\frac{3}{5}$ is less than log 5. Let $x = \frac{2}{3}$; then $10^{\frac{2}{3}} = \sqrt[3]{1000}$; but $5 = \sqrt[4]{625}$ ∴ $\frac{2}{3}$ is greater than log 5; log 5 is therefore between $\frac{2}{3}$ and $\frac{3}{4}$, or $\frac{8}{12}$ and $\frac{9}{12}$, that is between ·666666 and ·749999. This number is afterwards found to be ·698970. *See Art.* 205.

† When we have such a number as —3+·726401, it might appear more natural actually to incorporate the two parts, which would

197. III. The *modulus* of a system of logarithms is a constant multiplier, by means of which, when the logarithm of a number in one system is known, its logarithm in another system may be found.

198. *Schol.* 1. It follows from Art. 35, 41, that the logarithm of a product is equal to the sum of the logarithms of its factors; and that the logarithm of a quotient is equal to the difference of the logarithms of the dividend and divisor: that is, $\log abc = \log a + \log b + \log c$; and $\log \frac{a}{b} = \log a - \log b$.

Also, from Art. 88, 94, it follows, that the logarithm of the nth power of a number is n times the logarithm of the number; and the logarithm of the nth root, the nth part of the logarithm of the number: or that $\log b^n = n \log b$; $\log b^{\frac{1}{n}} = \frac{1}{n} \log b$; and $\log b^{\frac{m}{n}} = \frac{m}{n} \log b$.

199. *Cor.* Hence, it is plain that the subtraction of 1 from any logarithm, gives the logarithm of the quotient obtained by dividing the number by 10; the subtraction of 2, the logarithm of the quotient obtained by dividing the number by 100, &c.; and, therefore, that the logarithms of decimals will differ from the logarithms of integers composed of the same figures, only in the characteristic. It is this property of 10 which makes it the most convenient base.

give $-2{\cdot}273599$; but the mode of notation explained in the text, enables us to make the same logarithm serve for all numbers, integral, fractional, and mixed, which consist of the same figures, as will appear in Art. 199.

200. *Schol.* II. If there be a series of quantities in geometrical progression, their logarithms will be in arithmetical progression.

Let b, cb, c^2b, c^3b, &c. be a geometrical progression; and let x, x', be the logarithms of b and c: then (Art. 198) the progression is the same as

$$a^x,\ a^{x+x'},\ a^{x+2x'},\ a^{x+3x'},\ \&c.$$

in which the logarithms

$$x,\ x+x',\ x+2x',\ x+3x',\ \&c.$$

are in arithmetical progression.

201. PROB. I. *To expand* a^x *in a series according to the powers of* x.

Assume $a^x = h+kx+lx^2+mx^3+nx^4+$ &c.

squaring both sides we have

$$a^{2x} = h^2+2hkx+k^2 \left|\begin{array}{l} +2hl \end{array}\right| x^2+2hm \left|\begin{array}{l} +2kl \end{array}\right| x^3+\ l^2 \left|\begin{array}{l} +2km \\ +2hn \end{array}\right| x^4+\ \&c. \qquad (1)$$

also putting $2x$ for x in the first series we obtain

$$a^{2x} = h+2kx+4lx^2+8mx^3+16nx^4+\ \&c. \qquad (2)$$

Hence (Art. 149) equating the coefficients of the like terms, we find

$h^2=h$, or $h=1$; $2hk=2k$ or $k=k$; $k^2+2hl=4l$, or $l=\frac{k^2}{1.2}$; $2hm+2kl=8m$, or $m=\frac{k^3}{1.2.3}$

Substituting in the original series these values of the coefficients, we find

$$a^x = 1+kx+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}+\frac{k^4x^4}{1.2.3.4}\ \&c. \qquad (3)$$

In this k is yet unknown; to find it, put $a=1+b$, and we then have

$$a^x=(1+b)^x=1+xb+\frac{x(x-1)}{1.2}b^2+\frac{x(x-1)(x-2)}{1.2.3}b^3+\&c.$$

or, if the multiplications indicated be performed, and M, N, &c. be put for the coefficients, which x^2, x^3, &c. then have,

$$a^x=1+\left\{b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.\right\}x+Mx^2+Nx^3+\&c.$$

As this value of a^x must be identical with the former, we have, by equating the coefficients, and since $b=a-1$,

$$k=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.$$

If we represent by e that value of a which makes $k=1$ we shall have

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\&c.$$

and if in this series we put $x=1$ we find

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\&c.$$

or, by actual calculation of the values of the several terms,

$$e=2{\cdot}718281828459....$$

202. *Schol.* A power having an unknown quantity for its index, is called an *exponential* quantity; and the formula, series (3) last Art., giving the development of a^x is usually known by the name of *the exponential theorem:* it is of great importance in analysis.

203. II. *To find the logarithm of a number.*

Let N be any number, and x its logarithm to the base a, so that $a^x=N$: then it is required to determine x.

Raising both sides to the power whose index is y, we have

$$a^{xy}=N^y$$

Expanding both sides by the exponential theorem we find

$$1+\frac{kxy}{1}+\frac{k^2x^2y^2}{1.2}+\&c.=1+\frac{k'y}{1}+\frac{k'^2y^2}{1.2}+\frac{k'^3y^3}{1.2.3}+\&c.$$

where

$$k=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\&c.$$

and

$$k'=(N-1)-\tfrac{1}{2}(N-1)^2+\tfrac{1}{3}(N-1)^3-\tfrac{1}{4}(N-1)^4+\&c.$$

as proved before.

Equating the coefficients of y on both sides of the above equation, we have

$$kx=k', \text{ and } x=\frac{k'}{k}$$

We have, therefore,

$$\log N=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4\ \&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4\ \&c.}$$

The numerator of this fraction is independent of the base a, and will be the same in every system of logarithms; but the value of the denominator will depend upon a, and will be different in different systems. It is the reciprocal of this denominator which is called the ***modulus*** of the system: putting the modulus $= M$, and $N = 1+n$ we have

$$\log(1+n)=M\left(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4\ \&c.\right)$$

and therefore,

$$\log(1-n)=M\left(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4\ \&c.\right)$$

Hence, by subtraction, and Art. 198, we have

$$\log(1+n)-\log(1-n)=\log\frac{1+n}{1-n}=2M\left(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5\&c.\right)$$

Now, since in this equation n may have any value, let

$$n=\frac{1}{2N+1};\ \text{then } 1+n=\frac{2N+2}{2N+1};$$

and $1-n=\frac{2N}{2N+1}$; therefore, $\frac{1+n}{1-n}=\frac{N+1}{N}$

Hence we have, by substitution, and Art. 198, and transposing log N.

$$\log(N+1)=\log N+2M\left(\frac{1}{2N+1}+\frac{1}{3(2N+1)^3}+\frac{1}{5(2N+1)^5}+\frac{1}{7(2N+1)^7}\,\&c.\right)$$

This formula will give the logarithm of any number, that of the next less number being known.

204. The modulus M may have any value. In Napierian logarithms, it is taken equal to unity: for calculating these, therefore, we have the general formula

$$\log(N+1)=\log N+2\left(\frac{1}{2N+1}+\frac{1}{3(2N+1)^3}+\frac{1}{5(2N+1)^5}+\frac{1}{7(2N+1)^7}\,\&c.\right)$$

in which N must be taken, successively, equal to the natural numbers 1, 2, 3, 4, &c. We thus find

$$\log 2=2\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\cdot3^5}+\frac{1}{7.3^7}\,\&c.\right)=\cdot693147.$$

$$\log 4=2\log 2=1\cdot386294.$$

$$\log 5=2\left(\frac{1}{9}+\frac{1}{3\cdot9^3}+\frac{1}{5\cdot9^5}\,\&c.\right)+\log 4=1\cdot609438.$$

$$\log 10=\log 2+\log 5=2\cdot302585.$$

&c. &c.

205. III. *The logarithm of a number in one system being given, to find its logarithm in any other system.*

Let N be a given number, and let y, x, be its logarithms in two systems whose bases are a, e; so that

$$a^y = N,\ e^x = N;$$

and let the logarithms of N in the two systems be denoted by $\log_e N$, and $\log_a N$ respectively; then $x = \log_e N$, and $y = \log_a N$. Now it is plain that

$$e^x = a^y;$$

and therefore, by taking the logarithms of both sides, to the base e

$$x = y \log_e a,$$

whence

$$y = \frac{x}{\log_e a};$$

or substituting the above values of x and y,

$$\log_a N = \frac{\log_e N}{\log_e a};$$

or

$$\log_a N = \log_e N \times \frac{1}{\log_e a}$$

This equation expresses the constant ratio which the logarithms of N have to each other, in the two systems.

206. Since the only systems in use are the common and Napierian, let

$$a = 10,\ e = 2{\cdot}718281828459\ldots.\ \text{(Art. 196)};$$

then we have

$$\log_a N = \frac{1}{\log_e 10} \times \log_e N;$$

that is, *the common logarithm of any number is equal to the Napierian logarithm of the same number, multiplied by the reciprocal of the Napierian logarithm of* 10: this multiplier, $\frac{1}{\log_e 10}$, is therefore the *modulus*.

Now $\log_e 10 = 2{\cdot}3025850929\ldots.;$

therefore, $M = \frac{1}{\log_e 10} = \cdot 4342944819\ldots,$

which is the modulus of the common system. The common logarithms may, therefore, be calculated either by substituting this value of M in the general formula of Art. 204, or by multiplying the Napierian logarithm by the modulus. The former method is, of course, by far the more direct and expeditious: in this way we have

$$\log 2 = \cdot 86858896\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5\cdot 3^5} \text{ \&c.}\right) = \cdot 301030.$$

$$\log 4 = 2 \log 2 = \cdot 601060.$$

$$\log 5 = \cdot 86858896\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} \text{ \&c}\right) + \log 4 = \cdot 698970.$$

$$\log 10 = \log 2 + \log 5 = 1\cdot 000000.$$

The logarithms of all numbers are calculated in the same manner; and thus a table of logarithms is formed, for the purpose of simplifying arithmetical calculations (Art 198). The method of using these is explained in books of tables.

APPLICATION OF LOGARITHMS.

1. Raise 9 to the ninth power.

Here $\log 9^9 = 9 \log 9$;

but $\log 9 = \cdot 9542425$,

and $\cdot 9542425 \times 9 = 8\cdot 5881825 = \log 387420489 \therefore 9^9 = 387420489$.

2. Extract the 7th root of 67943.

Log $67943 = 4\cdot 8321447$; $\log \sqrt[7]{67943} = \frac{1}{7} \log 67943 = \cdot 6903064 = \log 4\cdot 901245 \therefore \sqrt[7]{67943} = 4\cdot 901245$.

3. Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x be the number; then

$$x = \frac{7^4 \times 5^5}{9^6} \therefore \log x = 4 \log 7$$

$+5 \log 5 - 6 \log 9 = \log 14{\cdot}12$ nearly, $\therefore x = 14{\cdot}12$ nearly.

4. From the equation $2^x = 1976$ find x.

Here $x \log 2 = \log 1976 \therefore x = \frac{\log 1976}{\log 2} = 10{\cdot}9$ nearly.

5. Given $a^{b^x} = c$; required x.

Let $$b^x = y,$$

so that

$$a^y = c;$$

then

$$x \log b = \log y$$

and

$$x = \frac{\log y}{\log b};$$

also

$$y \log a = \log c \therefore y = \frac{\log c}{\log a},$$

hence

$$x = \frac{\log (\log c) - \log (\log a)}{\log b}.$$

6. In the expression $s = \frac{ar^n - a}{r-1}$ (Art. 180)

required n.

Clearing and taking the logs. of both sides, we have

$$\log a + n \log r = \log (s.\overline{r-1} + a)$$

hence

$$n = \frac{\log (s.\overline{r-1} + a) - \log a}{\log r},$$

an expression for the number of terms in a Geometric series.

EXERCISES.

1. Given $a^x = b$; find x. *Ans.* $x = \dfrac{\log b}{\log a}$.

2. Given $e^{bx} = c$; find x, *Ans.* $x = \dfrac{\log c}{b \log a}$.

3. Given $20^x = 100$; find x. *Ans.* $x = 1{\cdot}537$.

4. Given $\log 2 = {\cdot}301030$, find those of 25 and ·0125.

Ans. 1·397940 and $\bar{2}{\cdot}096910$.

5. From the equation $(\frac{4}{3})^x = 54\frac{1}{2}$, find x.

Ans. $x = 17{\cdot}917$.

6. Given $a^{mx} b^{nx} = c$; find x.

Ans. $x = \dfrac{\log c}{m \log a + n \log b}$.

7. From the logarithms given in the last Art. find those of 8, 32, 20, 50, and 40.

CHAP. XIII.—THEORY OF QUADRATIC EQUATIONS.

ART. 207. DEF I. Any number which, being substituted for the unknown quantity in an equation, renders its two sides identical, is called a *root* of the equation.

208. We have seen in Art. 144 that one general form of a quadratic equation is

$$x^2+px=q, \text{ or } x^2+px-q=0.$$

Now, if r be a root of this equation, that is, a value of x, we shall have by substitution,

$$r^2+pr=q;$$

whence

$$x^2+px=r^2+pr,$$

or

$$x^2-r^2+px-pr=0;$$

hence,

$$(x+r)(x-r)+p(x-r)=0;$$

that is,

$$(x-r)(x+r+p)=0:$$

from which last equation it follows, that, *if* r *be a root of the equation* $x^2+px=q$, *the left member is divisible by* x—r; and conversely, *if* x—r *divide the left member without remainder,* r *is a root of the equation.*

209. Since a product becomes zero when either of its factors becomes zero, $(x-r)(x+r+p)=0$ both when $x-r=0$, which gives $x=r$, and when $x+r+p=0$, which gives $x=-r-p$: hence, r being one root, $-r-p$ is the other; and *hence a quadratic has two roots.*

210, Since $x^2+px-q=(x-r)(x+r+p,)$ it follows

that *a quadratic equation is the product of two factors, which are of the first degree with respect to* x.

211. Adding together the two roots we have

$$r+(-r-p)=-p;$$

hence, *the sum of the roots of a quadratic equation is equal to the coefficient of the first power of the unknown with its sign changed.*

212. Since $r^2+pr-q=0$, transposing and changing all the signs we have

$$-r^2-pr=-q, \text{ or } r(-r-p)=-q;$$

and, therefore, *the product of the roots of a quadratic equation is equal to the absolute term with its sign changed.**

213 The general value of x (Art. 145,)

$$x=\frac{\mp b\pm\sqrt{(b^2\pm 4ac)}}{2a}$$

includes the four following forms:

1. 2. $\left.\begin{array}{l}\\\end{array}\right\}$ $ax^2\pm bx=+c$ ..... $x=\left\{\begin{array}{l}\dfrac{-b\pm\sqrt{(b^2+4ac)}}{2a},\\ \text{or}\\ \dfrac{b\pm\sqrt{(b^2+4ac)}}{2a}.\end{array}\right.$

3. 4. $\left.\begin{array}{l}\\\end{array}\right\}$ $ax^2\pm bx=-c$ ..... $x=\left\{\begin{array}{l}\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a},\\ \text{or}\\ \dfrac{b\pm\sqrt{(b^2-4ac)}}{2a}.\end{array}\right.$

In the four values of the first two of these forms, since the quantities under the radical sign are both positive, the square root can be extracted either exactly or approxi-

* We might also deduce these properties immediately from the general values in Art. 144. The work is left as an exercise for the student.

mately, and therefore the roots of the equation are real: and since $\sqrt{(b^2+4ac)}$ is necessarily a greater number than b, the roots will be positive when the sign of the radical is $+$, and negative when the sign of the radical is $-$.

Again, in the four values of the last two of these forms, in order to render the evolution possible, and the roots real, we must have $b^2 > 4ac$; but if $b^2 < 4ac$, the roots are both imaginary: and since $\sqrt{(b^2-4ac)}$ is a less number than b, the roots will be negative when b is positive in the equation, and positive when b is negative.*

If we suppose $b^2 = 4ac$, or $c = \frac{b^2}{4a}$, in the equation $ax^2 \pm bx = -c$, the radical quantity vanishes, and the two values of x are in the one case $-\frac{b}{2a}$, and in the other $\frac{b}{2a}$; the roots are then equal to one another, and the equation is a square.

214. We might deduce these results from the principles established in Art. 211, 212; for if r, r', are the roots of the equation $x^2+px = q$, we have

$$r+r' = -p,\ rr' = -q;$$

or $$p = -(r+r'), \text{ and } q = -rr':$$

the roots have therefore different signs, since their product is negative. Also, since the sum of the roots is negative when p is positive, and positive when p is negative (Art. 211), that is, when $x^2-px = q$, it follows that the root which is the greater number has a different sign from p.

Again, in the equation $x^2 \pm px = -q$, q being negative,

* This Art. may be illustrated by examples taken from compound quadratics, p. 203, et seq.

the two roots will have the same sign, since their product is positive; but as their sum has always a different sign from p, they are both negative when p is positive, and both positive when p is negative.

215. If we compare the results to which we have arrived in the last Article with the signs of the terms of the general equation, we shall perceive that there is always a connexion between the number of positive roots, and the number of changes of the signs from + to —, or from — to +. Thus, in the equation $x^2+px+q=0$, since the root that is the greater number is always of a different sign from p, and since in this case the product of the roots is positive, the other root also must be negative: thus, *there is no positive root, and no change in the signs of the terms of the equation.*

Again, in the equation $x^2-px+q=0$, the root that is the greater number must be positive; but the product is positive, and therefore also the other root: there are then, two positive roots, and two changes of sign in the equation. And in the same manner it may be shown, that of the equations $x^2+px-q=0$, and $x^2-px-q=0$, having but one change of sign, *there is one root positive, and the other negative.*

216. From the general principles established in the preceding Articles, it follows,

1*st*, That when a quadratic is reduced to the form $x^2 \pm px = \pm q$, its roots may be found by inspection, whenever they are such as can be determined in finite terms.

2*d*, That when the roots of a quadratic are given, the equation will be formed by putting the difference between the unknown quantity and these roots equal to zero,

transposing all the terms to one side, and finding the product of the results.

3*d*, That the factors of a quadratic expression will be found by putting it equal to zero, and solving the equation that results.

EXAMPLES.

1. Find, by inspection, the roots of the equation

$$x^2-7x=44, \text{ or } x^2-7x-44=0.$$

Here it is required to find two numbers, whose product shall be —44, and sum +7: the numbers plainly are 11 and —4. We know from the last Art. also, that one root only is positive; and from Art 214, that that is the greater root.

2. Required the quadratic equation whose roots are 5 and —7.

Here $x=5$, $x=-7$; therefore, transposing, we have $x-5=0$, and $x+7=0$: multiplying, we find $x^2+2x-35=0$, or $x^2+2x=35$; which is the required quadratic.

3. Find the factors of $2x-15+x^2+a(3-x)$.

The roots are here found to be $a-5$, and 3; hence the factors are $x-a+5$, and $x-3$.

EXERCISES.

1. Required the factors of $pq^2+\frac{m^2-n^2}{p}-2mq$.

Ans. $q-\frac{m+n}{p}$, and $q-\frac{m-n}{p}$.

2. Required the factors of $(a+b)x+x^2-9x+(b-4)a+(4-b)5$. *Ans.* $x-4+b$, and $x-5+a$.

3. Required the factors of the expression

$$\frac{a}{a+2} - \frac{a-9}{3a-20} - \frac{9}{13}.$$

Ans. $a-11$, $a+54$.

4. Required the factors of $x^2-5x-6000$.

5. Required the quadratic whose roots are 10 and -9.

6. Required the quadratic whose roots are $\frac{1}{2}$ and $\frac{1}{3}$.

7. Required the factors of $a^2+3a-28$.

8. Required the factors of $2a^2c-3cx^2-b^2c-acx+bcx^2-a^2c$.

9. Required the quadratic whose roots are 7 and $-1\frac{1}{12}$.

10. Required the factors of $3400+111x-x^2$

CHAP. XIV.—SOLUTION OF GEOMETRICAL PROBLEMS.

ART. 217. A line may be represented by a number, expressing how many linear units the line contains. Thus, the line AB

A——————B

may be represented by any number, as 7; meaning that the line contains 7 inches, 7 feet, 7 miles, or 7 linear units of any kind. In the same manner, the line AB may be represented by any letter, as a; the letter being understood to denote a number of linear units. Also, if the two adjacent sides of a rectangle be represented by any numbers or letters, the product of those numbers or letters will express the number of superficial units in its area: thus, if a and b represent the adjacent sides, ab will represent the area. Hence, if a be the side of a square, its area will be represented by a^2. In the same manner, if a, b, c, represent the number of linear units in the length, breadth, and thickness, respectively, of a solid, the number of its solid units will be expressed by the product abc. Conversely, any single letter may be represented by a line, the product of two letters by a rectangle, and the product of three letters by a solid.

A rectangle *may*, however, be represented by a single letter as a; and if one side of such rectangle be represented by x, then the other will be $\frac{a}{x}$. Also if a represent the area of a square, $\sqrt{a}$ will represent the side. Conversely, it is not *necessary* that every product of two

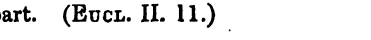
factors should be geometrically represented by a rectangle, and every product of three factors, by a solid: for ab may represent a line, which is a times the line b, that is, which contains a times as many linear units; and abc may express a rectangle, which is a times the rectangle bc. And as geometry is only concerned with the three dimensions of length, breadth, and thickness, it is plain, if a product consist of $3+n$ factors, they cannot all denote lines; and n of them, at least, must be taken as forming a coefficient to the geometrical magnitude expressed by the others.

The directions that have been given in Art. 148, for the solution of algebraic problems, are applicable also to the solution of geometrical problems. In order to translate the conditions of a problem into equations, the problem must be supposed to be resolved, and relations sought out between the given and required parts: for the purpose of extending these relations, and discovering new ones, additional lines may be drawn, or figures constructed. For this, however, or for the method of translating the conditions into algebraic language, no general rule can be laid down.

PROBLEMS.

1. To divide a straight line, so that the rectangle contained by the whole and one part, shall be equal to the square of the other part. (EUCL. II. 11.)

Let $AB = a$, $AC = x$ $\therefore$ $BC = a - x$; then, by the question, $a(a-x) = x^2$; whence $x^2 + ax = a^2$. Hence we have (Art. 145)



$$x = -\frac{a}{2} \pm \sqrt{\frac{5a^2}{4}};$$

or $$x = -\frac{a}{2} \pm \frac{a\sqrt{5}}{2} = \frac{a(\pm\sqrt{5}-1)}{2}.$$

Here, if the sign + be taken, x is positive (Art. 213), and less than a; but if the sign — be taken, it is negative, and greater than a: that is to say, the conditions are fulfilled, not only by a point C within the line, as the enunciation contemplated, but also by a point C′ on its continuation. The position of the point was supposed to be such that, in order to find it, something was to be taken from AB; whereas the point may also have such a position that, to find it, something must be added to AB (Art. 5,6). To answer this new condition, the question may be easily modified thus:

In a given straight line, or its continuation, to find a point such that the rectangle contained by the given line and the segment between one of its extremities and the required point, may be equal to the square of the segment between its other extremity and the same point.

If the value of x be put under the form $\sqrt{\left(a^2+\frac{a^2}{4}\right)}-\frac{a}{2}$, it is plain that we shall have (Eucl. II. 11)

$$\sqrt{\left(a^2+\frac{a^2}{4}\right)}-\frac{a}{2} = \text{EF}-\text{EA} = \text{AH}.$$

If a square be described on EF+EA = CF, instead of upon EF—EA = AF, and upon the opposite side from AG, BA produced will meet a side of the square in a point H′, which will give $\text{AB}.\text{BH}' = \text{AH}'^2$: this construction answers to the negative value.

If we had at first supposed the point to have also the position C′, and put BC, or BC′ = x, we should have had

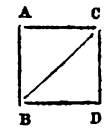
$AC = a - x$, $AC' = x - a$: then $(a-x)^2 = ax$, or $(x-a)^2 = ax$, either of which would give

$$x = \frac{3a}{2} \pm \sqrt{\frac{5a^2}{4}} = \frac{3a}{2} \pm \frac{a\sqrt{5}}{2};$$

both positive values.

2. Given the difference of the perimeter and diagonal of a square, to find the side and diagonal.

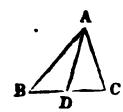
Let one side $AB = x$; then perimeter $= 4x$: also let $d =$ difference of perimeter and diagonal; then $BC = 4x - d$.



Then (Eucl. I. 47) $2x^2 = (4x-d)^2$; hence, $x = \frac{d}{4-\sqrt{2}} = \frac{d(4+\sqrt{2})}{14}$ (Art. 117). But $BC = 4x - d = x\sqrt{2}$; therefore $BC = \frac{d(4+\sqrt{2})\sqrt{2}}{14} = \frac{d(2\sqrt{2}+1)}{7}$.

3. Given the line bisecting the vertical angle of a triangle, and the segments into which it divides the base, to find the sides.

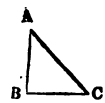
Let $AD = a$, $BD = b$, $DC = c$, $AB = x$, $AC = y$.



Then (Eucl. VI. 3) $b : c :: x : y$, $\therefore cx = by$, and $x = \frac{by}{c}$. Again (Eucl. VI. B), $xy = bc + a^2$, $\therefore x = \frac{bc+a^2}{y}$. Hence, $\frac{by}{c} = \frac{bc+a^2}{y}$; whence $y = \sqrt{\frac{bc^2+a^2c}{b}}$, and therefore $x = \frac{b}{c}\sqrt{\frac{bc^2+a^2c}{b}} = \sqrt{\frac{b^2c+a^2b}{c}}$.

4. In a right-angled triangle, given the difference of the

legs, and the difference of the hypotenuse and one leg, to find the hypotenuse and legs.



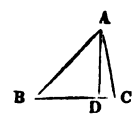
Let CB—BA $= a$, CA—AB $= b$, AB $= x$. Then $x^2 = (x+b)^2-(x+a)^2$, or $x^2+2(a-b)x=b^2-a^2$; whence (Art. 145),

$$x = \frac{2b-2a \pm \sqrt{(8b^2-8ab)}}{2} =$$

$b-a \pm \sqrt{(2b^2-2ab)} =$ AB; and therefore BC $= b \pm \sqrt{(2b^2-2ab)}$, and AC $= 2b-a \pm \sqrt{(2b^2-2ab)}$.

When the triangle is isosceles, $a = 0$, and $x = b\,(1 \pm \sqrt{2})$, which is obviously greater than the other value of x; and therefore, of all right-angled triangles upon the same hypotenuse, the isosceles is the greatest.

5. Given the perpendicular of a triangle, the difference of the segments of the base, and the ratio of the sides, to determine the triangle.



Let AD $= a$, BD—DC $= d$, and AB : AC :: 3 : 2. Also. let AB $= x$, AC $= y$, BC $= z$. Then, $x : y :: 3 : 2$; therefore $x = \frac{3y}{2}$, $y = \frac{2x}{3}$. Now (Eucl. VI. K),

$$dz = \left(x + \frac{2x}{3}\right)\left(x - \frac{2x}{3}\right) = x^2 - \frac{4x^2}{9}; \text{ hence, } x^2 = \frac{9\,dz}{5}.$$

Also (Eucl. I. 47),

$$x^2 = a^2 + \left(\frac{z+d}{2}\right)^2 = \frac{4a^2+d^2+2dz+z^2}{4} = \frac{9dz}{5}.$$

therefore, clearing of fractions, and transposing,

$$5z^2-26dz = -20\,a^2-5\,d^2,$$

which gives $z = \frac{26\,d \pm \sqrt{(676\,d^2-400a^2-100\,d^2)}}{10}$;

by reduction, $z = \frac{26\,d \pm 4\sqrt{(36d^2-25a^2)}}{10}$

Now, $x^2 = \frac{9\,dz}{5}$, therefore we have

$$x = \pm\sqrt{\frac{117d^2 \pm 18d\sqrt{(36d^2 - 25a^2)}}{25}} =$$

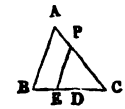
$$\pm\tfrac{3}{5}\sqrt{13\,d^2 \pm 2\,d^2\sqrt{(36d^2 - 25\,a^2)}}:$$

but $y = \frac{2}{3}x$, therefore

$$y = \pm\tfrac{2}{3}\sqrt{13\,d^2 \pm 2\,d^2\sqrt{(36\,d^2 - 25a^2)}}.$$

6. To bisect a triangle by a straight line drawn from a given point in one of its sides.

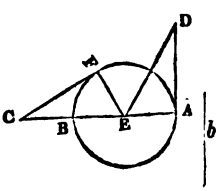
Let ABC be the given triangle, and P the given point; bisect BC in D, and make (Eucl. VI. 12) PC : CD :: AC : CE; then PC.CE = AC. CD = $\frac{1}{2}$ AC . CB, therefore AC . CB = 2PC . CE. But (Thomson's Eucl. VI. 23, *Note*) AC . CB : PC . CE :: ACB : PCE; therefore ABC = 2 PCE, or PE bisects the triangle.



Now, to determine the point E, let BC = a, AC = b CP = d, and CE = x; then $d : \frac{1}{2}a :: b : x$; hence $x = \frac{ab}{2d}$. When $d = \frac{1}{2}b$, $x = a$; or E coincides with B (Euc. I. 38).

7. To produce the diameter of a given circle, so that the tangent to the circle from the extremity of the produced part shall have a given ratio to a given line.

Let $m : n$ be the given ratio, and b the given line; from A draw AD perpendicular to AB, and determine AD by making $m : n :: b : \mathrm{AD} = b\,\frac{n}{m}$; join ED, and make EC = ED; draw the tangent CF, and join FE: then (Thomson's Eucl. I. 47,



Cor. 5) AD $= \mathrm{CF} = \frac{bn}{m}$ | Let now the radius $= r$, BC $= x$;

then $\mathrm{CF} = \frac{bn}{m} = \sqrt{(x+r)^2 - r^2}$; $\therefore x^2 + 2rx = \frac{b^2n^2}{m^2}$:

whence, $x = \frac{-2r \pm \sqrt{\left(4r^2 + 4\frac{b^2n^2}{m^2}\right)}}{2} = \frac{\pm\frac{2}{m}\sqrt{(r^2m^2 + b^2n^2)} - 2r}{2}$

$$= \frac{\pm\sqrt{(r^2m^2 + b^2n^2)} - rm}{m}.$$

If $m = n$, $\mathrm{CF} = b$, and $x = -r \pm \sqrt{(r^2 + b^2)}$; and if also $b = 2r$, $x = (\pm\sqrt{5} - 1)r$.

8. Given the base, b, and the altitude, a, of a triangle, to find the side of the inscribed square.

Ans. $\frac{ab}{a+b}$

9. Given the difference, d, between the diagonal of a square and one of its sides, to determine the square,

Ans. $(1+\sqrt{2})\,d =$ the side.

10. Given one side, a, of a right-angled triangle, and the sum of the hypotenuse and other side, b, to find the hypotenuse and other side.

Ans. Hyp. $= \frac{a^2+b^2}{2b}$, side $= \frac{b^2-a^2}{2b}$.

11. Given the sum, s, of the perimeter and diagonal of a square to find the side.

Ans. Side $= \frac{4s \pm s\sqrt{2}}{14}$.

12. Given the ratio, $m : n$, of the two sides of a triangle, and the segments, a and b, of the base, made by a perpendicular, to find the two sides.

Ans. $m\sqrt{\frac{a^2-b^2}{m^2-n^2}}$, $n\sqrt{\frac{a^2-b^2}{m^2-n^2}}$.

13. To divide a given straight line, a, so that the rectangle of the two parts may be equal to a given square, b^2.

Ans. The expression is $\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b^2\right)}$.

14. Given the breadth, a, of a street, and the perpendicular heights, b, c, of the houses on opposite sides, to find a point in the street, so that one extremity of a ladder being placed at it, the other will reach to the top of each house.

Ans. The distance of the point from the side of the street is, $\frac{a^2+b^2-c^2}{2a}$.

15. Given one side, a, of a right-angled triangle, and the alternate segment, b, of the hypotenuse, to find the whole hypotenuse and the other side.

Ans. Hyp., $\frac{b \pm \sqrt{(b^2+4a^2)}}{2}$; side, $\sqrt{\frac{b^2 \pm b\sqrt{(b^2+4a^2)}}{2}}$.

16. Given the hypotenuse, a, of a right-angled triangle, and the difference, d, between the other two sides, to find the sides.

Ans. Base $= \frac{1}{2}d \pm \frac{1}{2}\sqrt{(2a^2-d^2)}$; perp. $= -\frac{1}{2}d \pm \frac{1}{2}\sqrt{(2a^2-d^2)}$.

17. Through a given point in a given circle, to draw a chord of a given length, a.

The segments made by a diameter being x and $a-x$, we have $x = \frac{a \pm \sqrt{(a^2-4bc)}}{2}$.

18. In a rectangle given, a and b, the difference between the diagonal and each of the sides, to find the sides and diagonal.

Ans. Diag. $= a+b \pm \sqrt{2ab}$; the sides, $b \pm \sqrt{2ab}$, $a \pm \sqrt{2ab}$.

19. Given two sides, a and b, of a triangle, and the difference, d, of the segments of the base, made by a perpendicular, to find the base.

Ans. Base $= \frac{a^2-b^2}{d}$.

APPENDIX.

ANOTHER METHOD OF EXTRACTING THE CUBE ROOT.*

RULE.

(1) Point off the given number into periods of three figures in each, commencing at the units' place; (2) find the greatest cube in the first period, and place its cube root as the first figure of the answer; (3) subtract this cube from the first period, and bring down another period, or three zeros, if there be not another period; (4) in the column marked A, place three times the first figure of the root; (5) in the column B, place three times the square of this figure, and two dots to the right of the result; (6) find how often this number is contained in the result of (3) omitting the last two figures, and place this number as the trial figure of the root; (7) place it also to the right of the number in column A; (8) and multiply this number by the trial figure placing the unit figure of the result under the last dot; (9) add the result to the number above it; (10) and multiply the sum by the trial figure; (11) if this figure do not turn out too large, square it, and place the square under the last number in column B; but if it

* The following Rule is a particular case of a general method of solving equations, discovered by the late Mr. Horner. It is hoped, that as now given, it will be found to contain considerable improvements on the ordinary process. A full account of this method, and of the other recent discoveries in the solution of equations, will be found in an elaborate work by Professor Young, of Belfast, on the "Theory and Solution of Algebraic Equations."

be too large, erase the work, and try a less number; (12) add the result to the two numbers above it, and place two dots to the right of the result: this is the new trial divisor for the next figure of the root; (13) add twice the last figure of the root to the number already in column A, or else place in column A three times the portion of the root already found; (14) proceed as before to find the next figure; (15) after five or six figures have been obtained, the others required may be found by the ordinary rules for contracted division of decimals, using as divisor the number in column B, neglecting column A.

Exactly the same method is to be followed in Algebra, care being taken that the quantities be arranged according to the powers of some one letter.

EXAMPLES.

1. Required the cube root of 410172407.

A	B	
214	147..	410,172,407(743
8	856	343
2223	16556	67172
	16	62224
	16428..	4948407
	6669	4948407
	1649469	

2. Required the cube root of ·1 to 10 places of Decimals.

A	B	
3 times Root	3 times (Root)	
	48..	.100(.4641588833
126	756	64
	5556	36000
	36	33336
	6348..	2664000
1384	5536	2561344
	640336	102656000
	16	64602721
	645888..	38053279000
13921	13921	32311802375
	64602721	5741476625000
	1	5170534457312
	64616643..	570942167688
139235	696175	51706
	6462360475	5388
	25	5171
	6463056675..	217
1392458	11139664	193
	6463,16807164	24
		19
		5

3. Required the cube root of

$$8n^9 - 6n^8 - \tfrac{5}{2}n^7 - \tfrac{81}{8}n^6 + \tfrac{77}{12}n^5 + \tfrac{57}{12}n^4 + \tfrac{154}{27}n^3 - \tfrac{11}{6}n^2 - n - 1$$

A 3 times Root	B 3 times (Root)^2
$6n^3 - \frac{1}{2}n^2$	$12n^6$ $-3n^5 + \frac{1}{4}n^4$
	$12n^6 - 3n^5 + \frac{1}{4}n^4$ $\frac{1}{4}n^4$
$6n^3 - \frac{5}{2}n^2 - \frac{1}{3}n$	$12n^6 - 6n^5 + \frac{5}{4}n^4$ $-2n^4 + \frac{1}{2}n^3 + \frac{1}{9}n^2$
	$12n^6 - 6n^5 - \frac{5}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{9}n^2$ $\frac{1}{9}n^2$
$6n^3 - \frac{5}{2}n^2 - n - 1$	$12n^6 - 6n^5 - \frac{13}{4}n^4 + n^3 + \frac{1}{3}n^2$ $-6n^3 + \frac{5}{2}n^2 + n + 1$
	$12n^6 - 6n^5 - \frac{13}{4}n^4 - 5n^3 + \frac{11}{6}n^2 + n + 1$

$8n^9 - 6n^8 - \frac{5}{2}n^7 - \frac{81}{8}n^6 + \frac{77}{12}n^5 + \frac{57}{12}n^4 + \frac{134}{27}n^3 - \frac{11}{6}n^2 - n - 1$

$8n^9$ (Root $= 2n^3 - \frac{1}{2}n^2 - \frac{1}{3}n - 1$).

$-6n^8 - \frac{5}{2}n^7 - \frac{81}{8}n^6$

$-6n^8 + \frac{5}{2}n^7 - \frac{1}{8}n^6$

$-4n^7 - 10n^6 + \frac{77}{12}n^5 + \frac{57}{12}n^4 + \frac{134}{27}n^3$

$-4n^7 + 2n^6 + \frac{5}{12}n^5 - \frac{1}{6}n^4 - \frac{13}{27}n^3$

$-12n^6 + 6n^5 + \frac{13}{4}n^4 + 5n^3 - \frac{11}{6}n^2 - n - 1$

$-12n^6 + 6n^5 + \frac{13}{4}n^4 + 5n^3 - \frac{11}{6}n^2 - n - 1$

EXERCISES.

Given Cubes.	Cube Roots.
1. 405224.	*Ans.* 74.
2. $\frac{x^3}{8} + \frac{8}{27x^6} + \frac{2}{3x^3} + \frac{1}{2}$.	*Ans.* $\frac{x}{2} + \frac{2}{3x^3}$.
3. 1·092727.	*Ans.* 1·03.

4. $x^3 + \frac{8}{x^3} - 12x^2 - \frac{48}{x^2} + 54x + \frac{108}{x} - 112$. *Ans.* $x - 4 + \frac{2}{x}$.

5. Extract the cube root of 9 to 15 places of decimals.

Ans. 2·080083823051904.

NOTE A.

THE marginal note in page 1 goes upon the principle, that algebraic quantities always represent numbers. Respect for the distinguished mathematicians who think otherwise, requires that our reasons for this view should be stated.

Algebra has confessedly sprung up, by degrees, out of Arithmetic. It was first found convenient to have a symbol, by which an unknown number could be spoken of while the arithmetical process for finding it was going on. Next, since we have often occasion to speak of numbers in general, and since, in Arithmetic, each number is represented by a symbol or combination of symbols appropriated to itself, it was found that circumlocution would be saved by using arbitrary symbols to represent *classes of numbers* more or less extensive.

This use of general symbols to represent numbers, known or unknown, forms the essential distinction between Arithmetic and Algebra, and out of this all other differences arise. *First*, in Arithmetic, as well as in Algebra, operations are indicated by conventional marks: we may write $5+3$ and $5-3$ as well as $a+b$ and $a-b$. But in Arithmetic there is always at hand a preconcerted symbol, by which, and by no other, the result of an operation must be expressed: therefore we can always *perform* arithmetical operations; that is, we can always substitute the symbol of the result for the expression which indicates the operation, as 8 for $5+3$. In the investigations and even in the results of Algebra, on the contrary, we must rest satisfied with *indicating** operations, except in so far as the symbols of particular numbers may be introduced. Hence the signs which are often convenient to the Arithmetician, become indispensable to the Algebraist; no step of his processes can be taken without them, and his very results must be expressed by them: and it is the method of representing numbers by general symbols, which fixes the operator's attention on the meaning of the signs, and raises them to such a degree of prominence and importance. *Secondly*, in an arithmetical question, since the numbers concerned are particularized, we can modify the form of the problem (its substance being unchanged) so as to avoid a negative result; but when general symbols are used, the relative magnitudes of the numbers which they represent are kept out of view, and in whatever form the problem was at first stated, to that form the work and the answer must be

* This is obviously the case when we call $a+b$ the sum, and ab the product of a and b; and though the change of $a-(b+c)$ into $a-b-c$, and of $a(x+y)$ into $ax+ay$, or $a^3 \times a^2$ into a^5 is called *actual Subtraction* and *actual Multiplication*, yet it has dispensed but partially with signs of operation, though it disguises more or less the *origin* of the complex expressions on which it has been made.

strictly adapted; and hence we are often obliged to use the sign of Subtraction when there is nothing from which the quantity bearing that sign can be taken. Now, let $-\frac{ab}{4}$ be the answer to some algebraical question; then according to the first of the above observations, $-\frac{ab}{4}$ is merely a *requirement*, and according to the second, this requirement necessarily arises from our having adhered throughout the operations to the form in which the data were originally presented: or, putting the two observations together, $-\frac{ab}{4}$ means, that in order to fulfil the conditions of the problem *as stated*, a must be multiplied by b, the product divided by 4, and this result subtracted from 0. What is the meaning of this last demand? How shall we interpret this isolated negative quantity consistently with the original convention, which provided that — should denote Subtraction? The consideration of these queries leads to the views laid down in articles 1...6. Accordingly such views were put forward, though imperfectly, by the older writers on Algebra; but the paradoxical manner in which they spoke of quantities "less than nothing," seeming to use that phrase absolutely and not merely relatively, stirred up Mazeres and Frend to their well-known attacks on the whole doctrine of negative quantities. These authors were not able to banish the isolated $+a$ and $-a$ from Algebra; yet they succeeded in making subsequent writers adopt a different form of expression in speaking of such quantities: the sign — began to be spoken of as not always denoting Subtraction, but sometimes indicating that the quantity to which it is prefixed is taken in a "contrary sense"* to that which it would have, if it carried the sign +. The idea of two distinct meanings for each of the signs +, —, having thus gradually crept into Algebra, was at length formally propounded as the basis of certain important deductions, by M. Buée, in his ingenious "Memoire sur les Quantités Imaginaires," read before the Royal Society in June, 1805, and published in the Philosophical Transactions for 1806: his views will be best explained by the following abridged extract from his paper:—

"Chacun des signes + et — a deux significations tout-à-fait differentes. 1° Mis devant une quantité q ils peuvent designer deux *operations arithmetiques* opposées, dont cette quantité est le sujet. 2° Devant cette meme quantité ils peuvent designer deux *qualités* opposées ayant pour sujet les unités dont cette quantité èst composée.

"Dans l' Algebre ordinaire, c' est à dire, dans l' Algebre considerée comme l' Arithmetique universelle, une quantité isolée peut porter le signe +, qui dans ce cas n' ajoute rien à l' idée de cette quantité; mais elle ne peut pas porter le signe —. En

* This is one of those vague and indefinite phrases, by whose aid difficulties may be slurred over with an air of profundity.

effet, cette quantité etant supposé isolée, si on l' ajoute, ce ne peut être qu' á zero; si on la soustrait, ce ne peut être que de zero. Le premier est possible, mais le second est absurde. Par consequent, toutes les fois qu' on a, pour resultat d' une operation, une quantité precedée du signe—, il faut, pour que ce resultat ait un sens, y considerer quelque qualité. Alors l' Algebre ne doit plus être regardée simplement comme une Arithmetique universelle, mais comme une langue mathematique."

The same idea has been followed out with much boldness and originality by Dr. Peacock, (now Dean of Ely) in his "Treatise on Algebra," and in his "Report on the recent progress and present state of certain branches of Analysis," presented to the British Association for the Advancement of Science, in 1833. The two different aspects of Algebra which Buée had distinguished as *Algebre ordinaire* and *Algebre-langue*, Dr. Peacock regards as two distinct sciences, which he calls *arithmetical Algebra* and *symbolical Algebra*.

The following is an outline of our reasons for venturing to dissent from the eminent men who hold these opinions.

I. The principle on which these mathematicians go, in the interpretation of algebraic expressions, seems inconsistent with the nature of the connection between symbols and thought. A symbol derives its power of suggesting an idea from arbitrary association alone, and has no meaning whatever anterior to the convention by which such association is established. These observations apply to the spoken words of artificial language, as well as to the visible symbols of Algebra. It is perfectly true that we may, by a new convention, give the symbol a new signification; but in the same chain of reasoning, the same symbol must retain the same meaning, else no conclusion can be drawn. If, in translating the data of a problem into algebraic language, we use the sign — to denote subtraction, or rather to denote that subtraction is required, we must attach the same meaning to it through the work, and in the answers, else the conclusion is vitiated. That which is *given* in the enunciation concerning $-a$ as a quantity to be subtracted, cannot entitle us to say any thing concerning the $-a$ of the conclusion, taken in a sense *tout-a-fait different*. Yet it was in this way that isolated negative quantities were first forced on the attention of analysts: it follows, therefore, that an isolated negative quantity, *thus arising*, (and therefore any negative quantity), must either be interpreted so that its sign shall still denote subtraction, or be held void of all meaning. That the sign may have, in the conclusion, a different meaning from that which it had in the enunciation, M. Buée fairly avows.; most other writers of his school keep it out of view. To be consistent, M Buée ought to go the whole length of Mazeres and Frend, and maintain that such quantities always involve a contradiction.

Feeling, probably, the fallacy involved in M. Buée's notion, Dr. Peacock takes a different course, but one which is equally erroneous. According to what has been above stated, the nature of artificial symbols (including words) is such, that the mind must have possessed itself of some idea before it can invent a symbol.

The *definition* of a *symbol*, whether a word or an algebraic character, states the precise idea for which the symbol stands; and the *interpretation* of an *expression into which the symbol enters*, explains how the meaning of that expression is affected, by the introduction into it of a symbol with the given meaning: therefore, there can be no interpretation without a previous definition. Dr. Peacock does not perceive this; because, overlooking the true distinction between interpretation and definition, he supposes that one of them may replace the other. He assumes the existence of symbols, assumes the laws of their combination, and afterwards determines their meaning by what he calls interpretation. He treats words in the same manner; for example, he uses the word "incorporate" (Treatise on Algebra, Art. 7, 1st Ed.)* without attaching to it any definite idea. His intention is to employ it as a term of great generality, including arithmetical multiplication and division, with an indefinite number of other operations possessing some resemblance to them. Now, if there be any number of operations which possess something in common with arithmetical multiplication, while in other respects they differ from it, there can be no objection (unless it be one of expediency) to the use of a general term, by whose help all these operations may be grouped together. But, to assume, before-hand, such a connection among a number of unknown operations, and to give a general name to them when we cannot fix the generic character for whose sake they are formed into a class, is by no means allowable. A lively philosophical writer has happily explained the process of generalization, by saying, that "general terms are indorsements on the bundles of our ideas." But how can we write an indorsement, until we know accurately the contents of the bundle? In fact, the word "incorporate," so used, can have no meaning beyond that of the humble "multiply" of common Arithmetic. Dr. Peacock himself unconsciously testifies to this; for, after saying "symbols may be incorporated into one another, so as to reproduce a new quantity of the same or of a different kind," he is obliged, for the sake of being understood, to add "as in the operations of multiplication and division." The reader understands what is meant by "incorporation," as applied to these two operations (at least as applied to multiplication);† but has not, and cannot possibly have, any idea whatever attached to it, as including any other operations, until those operations themselves become known to him: till then, "incorporation" must, for him, be synonymous with "multiplication." After we have formed an idea of arithmetical multiplication, it is certainly conceivable that we may become acquainted with a similar operation not arithmetical, and that we may then frame a wider definition which shall include both of them, or apply to them in common a more

* In his second edition Dr. Peacock has abandoned the word here objected to, and has otherwise modified the expression of his views; but as he still abides by the distinction between arithmetical and symbolical algebra, no change has been made.

† Division is not properly incorporation, but rather decomposition.

general term. In like manner, we may successively include a third operation, and a fourth, and so on: but then the operations thus included must first have become known; and the extension of the definition, or the enlargement of the application of the term, so as to include unknown operations, is not only illogical but impossible. Reader and author may imagine that the words they use express some idea more general than that of ordinary multiplication; but they deceive themselves: it is not in the nature of the human mind to form a general idea in any other way, than by observing the common properties of known objects; but it is one of the most common illusions of the human mind, an illusion from which, perhaps, no man is wholly free, to fancy that it has grasped an idea when it is only playing with a word. The source of the error we are now combating, is the same with that which gave origin to the "intelligible species" of the schools; namely, the want of a clear and steadfast apprehension of the nature of general terms.

It would be an unnecessary repetition of what has been just stated, to point out that the same objection lies against operating with the signs $+$, $-$, or the marks $\times, \div$, or any other substitutes for words, until their meaning has been first defined. Neither a word nor an algebraic character deserves the name of symbol, unless there be attached to it a precise idea: it is but an empty sound or a useless blot on paper.

The expressions called *impossible* or *imaginary quantities* are to be explained on the same principle as negative quantities. $\sqrt{-a^2}$ is a *requirement*, arising from our having adhered throughout the operation that produced it, to the form in which the data were originally presented; that is to say, in order to fulfil the conditions of the question *as stated*, we must extract the square root of a negative quantity. What is the meaning of this demand? how shall we interpret it, consistently with the rules of involution as derived from the original convention which established the meaning of the negative sign? In the case of Algebra applied to Geometry, a satisfactory explanation has been obtained; $\sqrt{a^2}$ (page 293,) represents the side of that square whose area is represented by a^2; and since $-a^2$ means a square placed in a reversed position with regard to the square represented by a^2, it follows that $\sqrt{-a^2}$ denotes the side of this reversed square, that is, a line at right angles to the line denoted by $\sqrt{a^2}$.* Attempts have been made to assign a meaning to the square roots of negative quantities in other cases than that of their application to Geometry; but the views on which these attempts are founded do not appear sufficiently strict for mathematical purposes. A general principle of interpretation may be attainable, but has not yet been attained. Imaginary quantities, like vanishing fractions, require something besides the mere notation to determine their value.

II. A similar, though distinct error, is involved in Dr. Peacock's

* See Buée, Memoire, &c. *ut suprà;* Warren on the geometrical representation of the square roots of negative quantities; and Peacock's Algebra, Chap. XII. 1st Ed. and Chaps. XXV. and XXXI, 2nd Ed.

mode of *assuming* rules instead of *proving* them. The word *assume* is used by mathematicians in three senses; namely, to establish a convention, to lay down a hypothesis, to state an axiom. When Dr. Peacock "assumes" that the order of the factors in multiplication is indifferent, and that like signs produce + and unlike signs —, in which of these senses does he use the word "assume"? It cannot be in the second, for the thing assumed forms the *thesis* of his proposition: nor can it be in the first, for his proposition is an assertion, not an explanation. Now a convention refers to something which is to be *understood*, not *believed:* he must, therefore, have used the word "assume" in the third sense; that is to say, a convention has been established, giving a meaning to the signs +,—, and Dr. Peacock requires the learner to believe, without proof, that the rule in question is consistent with this meaning. Surely this cannot be regarded as a self-evident truth. It ought to have been shown, not only to be consistent with, but to arise out of, the definition of the signs. The fact is, that Dr. Peacock seems not to have distinguished between the second and the third senses, and to have imagined that he was only establishing a convention, when he was requiring an assent to a theorem.

Having stated these objections to the opinions of M. Buée and Dr. Peacock, we must now stand on the defensive. After stating in a few lines the views which he opposes, but which, he says, are sanctioned by "many even among the masters of algebraical science," Dr. Peacock proceeds to "reply to such opinions" by a series of counter-statements which express opinions of his own. The only one of these that requires notice is the following: "that it is an abuse of the term generalization to apply it to designate the process of mind by which we pass from the meaning of $a-b$ when a is greater than b, to its meaning when a is less than b; or from that of the product ab when a and b are abstract numbers, to its meaning when they are concrete numbers of the same, or of a different kind." (*3d Report Brit. Assoc. page* 194.) On $a-b$ enough has been already said; we must add a word on Multiplication. It has been fashionable among mathematical writers, for the last thirty years, to distinguish between abstract and concrete numbers; but it is not very clear that they have always understood accurately the meaning of those terms. Numbers are called abstract when the kind of units they denote is not specified, and concrete when it is: "abstract" never means (though some seem to think it does,) that the numbers so called have an independent existence. Thus, when we say "five times seven is thirty-five," or in symbols, $7 \times 5=35$, we have not before our minds any *particular kind* of units; but we must be aware that both multiplier and multiplicand mean units of *some kind:* 7 must mean seven *somethings*, and 5 means that those somethings are repeated or *taken* five times; but in this case it matters not what kind of things they are: 7 *any things* repeated 5 times will produce 35 things of the same kind with the seven things. It is plain that the multiplicand may be either abstract or concrete; but to which of these classes of

numbers does the multiplier belong? It always denotes *times* or *repetitions*, and might therefore be considered as concrete, since the things it denotes are always specified; if we understand it thus, all multipliers are concrete, and the case which Dr. Peacock supposes of both factors being abstract never can occur. But as the idea expressed by "times," or "repetitions," is itself highly abstract, and as the quantity produced by such repetition must depend on the kind of units expressed by the multiplicand, the multiplier is considered an abstract number: all mathematicians call it so when it expresses repetitions of the multiplicand; and we have just shown that it never expresses any thing else. Therefore, the case which Dr. Peacock supposes, of both factors being concrete, never can occur. We often speak, it is true, of multiplying concrete quantities together; but such expressions must be understood as mere abbreviations, else they are perfectly nonsensical. Thus, to find the interest of a sum of money for any number of days, the common rule is, "Multiply the principal by the days, and the product by twice the rate, and divide by 73,000." Here, "multiply by the days," can have no other meaning than "repeat this product as often as there are days;" so that the multiplier now means not units of time, but *repetitions* of the multiplicand. And when the area of a rectangle, whose sides contain 7 and 5 linear units respectively, is expressed by the product, 7×5, the factors no longer express linear units. The multiplicand denotes 7 superficial units, 7 little squares described on the linear units in one side, and the multiplier denotes that this number of superficial units must be 5 times repeated to find the number of superficial units in the entire rectangle. The rule for finding the area, if stated at full length, would therefore be, "Take as many superficial units as there are linear units in one side, and repeat this number of superficial units as often as there are linear units in the other side." The practical rule, "Multiply the adjacent sides together," can be understood in no other sense than as an abbreviation of the above, founded on the obvious principle, that when the rectangle is divided into rows of superficial units by lines parallel to its sides, there must be as many of such rows as there are linear units in one side, and as many superficial units in each row as there are linear units in the other side. Therefore, using the words "abstract" and "concrete," as they are constantly used by mathematicians, without raising any metaphysical question as to the correctness of the distinction, it appears, that the multiplier is always an abstract number; that the multiplicand may be either abstract or concrete; and that the product always denotes units of the same kind with those denoted by the multiplicand. It appears further, that an algebraical product of two factors, as ab, representing a rectangle, is to be understood as *one numerical symbol*, expressing the superficial units in the area of the rectangle: also, that a and b, considered as factors of this product, no longer represent lines; one of them denotes superficial units, and the other is an abstract number. To speak of multiplying or incorporating lines or their symbols algebraically, in any other than

this strict arithmetical sense, is to deliver our understandings into captivity to a jargon of unmeaning words.

We have spoken thus fully on multiplication, because we are anxious to guard the student against an illusion which has been the source of many errors, and by which many able mathematicians have been led astray. Using Algebra chiefly in its application to Geometry, and operating on the symbols of the one science while the realities of the other were before their minds, the early analysts established a direct association between the algebraic character and the geometrical magnitude, leaving out of view the number, through which alone they have any real connection. Hence the confusion of arithmetical and geometrical language so common in these writers; as when they spoke of the *product* of two lines, the *rectangle* of two abstract numbers, and the *square* of a sum of money. Mr. Frend wished to correct these abuses: he was quite right in maintaining that algebraic quantities always represent numbers, and that algebraic signs always indicate arithmetical operations; but he was wrong in concluding from this, that isolated negative quantities can have no meaning. Unfortunately, he did not give much prominence to the former doctrine, and dwelt largely on the latter; so that subsequent writers adopted his error, and neglected what was sound in his views. Thus arose a new school of algebraists, differing from the old in this, that they are more consistent in error. The older writers had spoken of the *signs* as always denoting arithmetical operations, but had not regarded the *quantities* as always expressing numbers: the new school maintain, with Frend, that isolated negative quantities cannot be interpreted in accordance with arithmetical principles; but, unwilling to render Algebra the powerless thing which it would be without such quantities, they sacrifice the philosophy of the science to its algorithm, and hold that neither its symbols nor their signs are always arithmetical. Dr. Peacock has attempted to remodel the whole science in conformity with this view. He has failed; but his failure does not lessen our respect for his great talents, nor detract any thing from his well-earned mathematical fame. His errors are not so much mathematical, as metaphysical; and it is probable that his neglect of verbal accuracy, and the consequent obscurity of his diction, have helped to conceal them even from himself. We cannot but admire the mathematician, and his perfect knowledge of the laws of combination of symbols; though we cannot but perceive that he has not paid sufficient attention to the laws of their connection with thought, or to the laws of thought itself.

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